Result: Primal Variational Formulation
$\hat{=}$ principle of virtual displacement (work)

(2) Find $u \in V_0 = \{ v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_u \}$:

$$a(u,v) = \langle F, v \rangle \quad \forall v \in V_0,$$

where

$$\langle F, v \rangle := \int_\Omega f_i v_i \, dx + \int_{\Gamma_t} t_i v_i \, ds = \int_\Omega f^T v \, dx + \int_{\Gamma_t} t^T v \, ds$$

$$a(u,v) := \int_\Omega \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx = \int_\Omega \sigma(u) \cdot \varepsilon(v)\, dx$$

$$\overset{\text{HOOK}}{=} \int_\Omega D_{ijkl}\, \varepsilon_{kl}(u)\, \varepsilon_{ij}(v)\, dx = \int_\Omega \varepsilon^T(u)\, \underset{\substack{\uparrow \\ \text{matrix of} \\ \text{elastic constants}}}{D}\, \varepsilon(v)\, dx$$

$$\underset{\text{(mms)}}{\overset{\text{isotropic}}{=}} \int_\Omega \Big\{ \lambda \underbrace{\sum_{k=1}^{3} \varepsilon_{kk}(u)}_{\operatorname{div} u} \underbrace{\sum_{l=1}^{3} \varepsilon_{ll}(v)}_{\operatorname{div} v} + 2\mu \sum_{i,j=1}^{3} \varepsilon_{ij}(u)\, \varepsilon_{ij}(v) \Big\} dx$$

$$= \int_\Omega \{ \lambda \operatorname{div} u \cdot \operatorname{div} v + 2\mu\, \varepsilon^T(u)\, \varepsilon(v) \}\, dx$$

with given

- $f = (f_1, f_2, f_3)^T \in [L_2(\Omega)]^3$, $t = (t_1, t_2, t_3)^T \in [L_2(\Gamma_t)]^3$,
- $D_{ijkl} \in L_\infty(\Omega)$: $0 < \lambda_{min}(D) \le \lambda(D(x)) \le \lambda_{max}(D)$ a.e. in Ω,
  resp. $\lambda, \mu \in L_\infty(\Omega)$: $0 < \underline{\lambda} \le \lambda(x) \le \overline{\lambda}$, $0 < \underline{\mu} \le \mu(x) \le \overline{\mu}$ -"- .

PVF $\equiv$ Minimization Problem ($a(\cdot,\cdot)$ sym. and $a(v,v) \geq 0$)

(3) Find $u \in V_0$: $J(u) = \inf_{v \in V_0} J(v)$,

with

$$J(v) = \underbrace{\frac{1}{2} \int_\Omega \sigma_{ij}(v)\, \varepsilon_{ij}(v)\, dx}_{\text{deformation energy}} - \underbrace{\Big(\int_\Omega f_i v_i \, dx + \int_{\Gamma_t} t_i v_i \, ds \Big)}_{\text{potential energy of ext. forces}}$$