3.2. Variational Problems in Elasticity

3.2.1. The Primal Variational Formulation (= pure displacement method)

- **Derivation of the Primal Variational Formulation:** (see Lectures on NuPDE and NuEPDE)

① $V_0 = \{ v = (v_1, v_2, v_3)^T \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_u \}$

② $-\int_\Omega \sigma_{ji,j} v_i \, dx = \int_\Omega f_i v_i \, dx \quad \forall v \in V_0$

$\parallel$

③ $\int_\Omega \sigma_{ji} v_{i,j} \, dx - \int_\Gamma \sigma_{ji} n_j v_i \, ds = \int_\Omega f_i v_i \, dx \quad \forall v \in V_0$

$\longrightarrow \sigma_{ji} v_{i,j} = \frac{1}{2}(\sigma_{ji} v_{i,j} + \sigma_{ij} v_{j,i})$ "σ_{ji} (equilibrium of momentum)"

$= \sigma_{ji} \frac{1}{2}(v_{i,j} + v_{j,i}) = \sigma_{ji} \varepsilon_{ji} = \sigma_{ij} \varepsilon_{ij}$

$\int_\Omega \sigma_{ij}(u) \varepsilon_{ij}(v) \, dx - \int_\Gamma \sigma \cdot n \cdot v \, ds = \int_\Omega f \cdot v \, dx \quad \forall v \in V_0$

④ $\int_\Gamma \sigma \cdot n \cdot v \, ds = \int_\Gamma \sigma_{ji} n_j v_i \, ds = \int_{\Gamma_u} \sigma_{ji} n_j \cdot \underbrace{v_i}_{=0} \, ds + \int_{\Gamma_t} \underbrace{\sigma_{ji} n_j}_{= t_i} \cdot v_i \, ds$

$= \int_{\Gamma_t} t_i v_i \, ds = \int_{\Gamma_t} t \cdot v \, ds$

⑤ $V_g = \{ v \in V : v = \bar{u} \text{ on } \Gamma_u \} = V_0$

$\bar{u} \in [H^{1/2}(\Gamma_u)]^3$ given

$\bar{u} = 0$ (WLG)