Example 2.21:

Let A be a $n \times n$ SPD matrix. We consider the special case $(B = A^{1/2},\ C = \mathbb{O},\ n = m)$:

$$\begin{bmatrix} A & A^{1/2} \\ A^{1/2} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{g} \end{bmatrix}.$$

Let us take

$$A_0 = (1-\alpha) A \quad \text{with some } \alpha \in (0,1),$$

i.e. $A - A_0 = \alpha A$.

Then

$$T = \frac{1}{1-\alpha} \begin{bmatrix} I & A^{-1/2} \\ \alpha A^{1/2} & I \end{bmatrix} \quad \text{and}$$

$$\tilde{T} = I = \begin{bmatrix} I & \mathbb{O} \\ \mathbb{O} & I \end{bmatrix}.$$

Consider the eigenvalue problem

$$T \underline{X} = \lambda \underline{X}$$

$$\det[T - \lambda I] = \det \begin{bmatrix} \left(\frac{1}{1-\alpha} - \lambda\right) I & \frac{1}{1-\alpha} A^{-1/2} \\ \frac{\alpha}{1-\alpha} A^{1/2} & \left(\frac{1}{1-\alpha} - \lambda\right) I \end{bmatrix}$$

$$= \left(\frac{1}{1-\alpha} - \lambda\right)^2 - \frac{\alpha}{(1-\alpha)^2} = 0, \quad \lambda = \frac{1 \pm \sqrt{\alpha}}{1-\alpha},$$

i.e. $\underline{\gamma} = \dfrac{1-\sqrt{\alpha}}{1-\alpha}$ and $\overline{\gamma} = \dfrac{1+\sqrt{\alpha}}{1-\alpha}$.

Remark 2.23:

1. Example 2.22 shows that $\overline{\gamma}$ is sharp!
2. $\underline{\gamma}$ can be improved, namely $\underline{\gamma} = \frac{1-\sqrt{\alpha}}{1-\alpha}$ that is now sharp too!