• We start by proving the estimate from below in (63):

$$(65)\quad [\tilde{T}\underline{x},\underline{x}] = ((A-A_0)\underline{u},\underline{u}) + ((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})$$

$$\le (1+\varepsilon)((A-A_0)\underline{u}_0,\underline{u}_0) + \left(1+\frac{1}{\varepsilon}\right)((A-A_0)\underline{u}_H,\underline{u}_H) + ((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})$$

ε-inequality
$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$
$\forall a,b \in \mathbb{R}\ \forall \varepsilon > 0$

Combining (59) $0 < A - A_0 \le \alpha A$ and (i) gives

$$(66)\quad ((A-A_0)\underline{u}_H,\underline{u}_H) \le \alpha(A\underline{u}_H,\underline{u}_H) = \alpha(BA^{-1}B^T\underline{\Delta},\underline{\Delta})$$

$$\le (1+\varepsilon)((A-A_0)\underline{u}_0,\underline{u}_0) + \left(1+\left(1+\frac{1}{\varepsilon}\right)\alpha\right)(BA^{-1}B^T\underline{\Delta},\underline{\Delta}) + (C\underline{\Delta},\underline{\Delta})$$

$$\le (1+\varepsilon)\underbrace{((A-A_0)\underline{u}_0,\underline{u}_0)}_{\le\, \left[T\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix},\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix}\right]} + \left(1+\left(1+\frac{1}{\varepsilon}\right)\alpha\right)\underbrace{((BA^{-1}B^T+C)\underline{\Delta},\underline{\Delta})}_{\overset{(ii)}{=}\left[T\begin{pmatrix}\underline{u}_H\\ \underline{\Delta}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\Delta}\end{pmatrix}\right]}$$

$$\left[T\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix},\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix}\right] = ((A-A_0)A_0^{-1}A\underline{u}_0,\underline{u}_0) = \underbrace{((A-A_0)A_0^{-1}(A-A_0)\underline{u}_0,\underline{u}_0)}_{\ge 0} + ((A-A_0)\underline{u}_0,\underline{u}_0)$$

$$\le \underbrace{(1+\varepsilon)}_{=}\left[T\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix},\begin{pmatrix}\underline{u}_0\\ 0\end{pmatrix}\right] + \underbrace{\left(1+\left(1+\frac{1}{\varepsilon}\right)\alpha\right)}_{=}\left[T\begin{pmatrix}\underline{u}_H\\ \underline{\Delta}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\Delta}\end{pmatrix}\right]$$

$$(1+\varepsilon) = 1+\left(1+\frac{1}{\varepsilon}\right)\alpha \iff \varepsilon^2 - \alpha\varepsilon - \alpha = 0$$

$$\iff \varepsilon_{1,2} = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4}+\alpha}, \text{ i.e.}$$

$$\boxed{\varepsilon = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4}+\alpha}}$$

$$= \underbrace{\left(1+\frac{\alpha}{2}+\sqrt{\frac{\alpha^2}{4}+\alpha}\right)}_{=:\ \underline{\gamma}^{-1}}[T\underline{x},\underline{x}]$$