PROOF:

- $[T\underline{X},\underline{Y}] = (S\underline{X},\underline{Y}) \overset{S=S^T}{=} (\underline{X}, S\underline{Y}) = [\underline{X}, T\underline{Y}] \ \forall \underline{X},\underline{Y}\in\mathbb{R}^n\times\mathbb{R}^m$

- $[T\underline{X},\underline{Y}] = (S\underline{X},\underline{Y}) =$

$$= \left(\begin{bmatrix} (A-A_0)A_0^{-1}A\underline{u} + (A-A_0)A_0^{-1}B^T\underline{\lambda} \\ BA_0^{-1}(A-A_0)\underline{u} + (BA_0^{-1}B^T+C)\underline{\lambda}\end{bmatrix}, \begin{bmatrix}\underline{v}\\ \underline{\mu}\end{bmatrix}\right)$$

$$= ((A-A_0)A_0^{-1}A\underline{u},\underline{v}) + ((A-A_0)A_0^{-1}B^T\underline{\lambda},\underline{v}) + (BA_0^{-1}(A-A_0)\underline{u},\underline{\mu}) + ((BA_0^{-1}B^T+C)\underline{\lambda},\underline{\mu})$$

- $[T\underline{X},\underline{X}] = (S\underline{X},\underline{X}) = ((A-A_0)A_0^{-1}A\underline{u},\underline{u}) + 2((A-A_0)A_0^{-1}B^T\underline{\lambda},\underline{u}) + ((\downarrow)\underline{\lambda},\underline{\mu})$

$[\tilde{T}\underline{X},\underline{X}] = (\tilde{S}\underline{X},\underline{X}) = ((A-A_0)\underline{u},\underline{u}) + ((BA_0^{-1}B^T+C)\underline{\lambda},\underline{\lambda})$

- Let us first decompose $\mathbb{R}^n\times\mathbb{R}^m$ by the splitting

(64) $\mathbb{R}^n\times\mathbb{R}^m \ni \underline{X} = \begin{pmatrix}\underline{u}\\ \underline{\lambda}\end{pmatrix} = \begin{pmatrix}\underline{u}_0\\ \mathbb{0}\end{pmatrix} \oplus \begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}$, $\quad \oplus\ [T\cdot,\cdot] = (S\cdot,\cdot)$

where $\underline{u}_H\in\mathbb{R}^n$: $A\underline{u}_H + B^T\underline{\lambda} = \mathbb{0}$ ($\exists!$ since A is SPD).
Then it is straightforward to verify that

(i) $(A\underline{u}_H,\underline{u}_H) = (BA^{-1}B^T\underline{\lambda},\underline{\lambda})$

(ii) $\left[T\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}\right] = \left(S\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}\right) =$

$$= ((A-A_0)A_0^{-1}A\underline{u}_H,\underline{u}_H) + 2((A-A_0)A_0^{-1}\underbrace{B^T\underline{\lambda}}_{=-A\underline{u}_H},\underline{u}_H) + (BA_0^{-1}B^T\underline{\lambda},\underline{\lambda}) + (C\underline{\lambda},\underline{\lambda})$$

$$= -((A-A_0)A_0^{-1}A\underline{u}_H,\underline{u}_H) + (BA_0^{-1}B^T\underline{\lambda},\underline{\lambda}) + (C\underline{\lambda},\underline{\lambda})$$

$$= -(A_0^{-1}\underbrace{A\underline{u}_H}_{=-B^T\underline{\lambda}},\underbrace{A\underline{u}_H}_{=-B^T\underline{\lambda}}) + \underbrace{(A\underline{u}_H,\underline{u}_H)}_{(i)=(BA^{-1}B^T\underline{\lambda},\underline{\lambda})} + (BA_0^{-1}B^T\underline{\lambda},\underline{\lambda}) + (C\underline{\lambda},\underline{\lambda})$$

$$= (BA^{-1}B^T\underline{\lambda},\underline{\lambda}) + (C\underline{\lambda},\underline{\lambda})$$

(iii) $\left[T\begin{pmatrix}\underline{u}\\ \underline{\lambda}\end{pmatrix},\begin{pmatrix}\underline{u}\\ \underline{\lambda}\end{pmatrix}\right] = \left[T\begin{pmatrix}\underline{u}_0\\ \mathbb{0}\end{pmatrix},\begin{pmatrix}\underline{u}_0\\ \mathbb{0}\end{pmatrix}\right] + \left[T\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}\right]$

$\left[T\begin{pmatrix}\underline{u}_0\\ \mathbb{0}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}\right] = \left(S\begin{pmatrix}\underline{u}_0\\ \mathbb{0}\end{pmatrix},\begin{pmatrix}\underline{u}_H\\ \underline{\lambda}\end{pmatrix}\right) = 0$ (mms)