2.4.2. The ARROW-HURWICZ-Algorithm

- The preconditioned Arrow-Hurwicz-Algorithm for solving system (46) $\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{g} \end{bmatrix}$

(54)
$$\boxed{\begin{array}{l} \text{Initial guess: } \underline{u}_0 \in \mathbb{R}^n, \ \underline{\lambda}_0 \in \mathbb{R}^m \text{ given} \\ \text{Iteration: } k = 0, 1, \ldots \\ G \dfrac{\underline{u}_{k+1} - \underline{u}_k}{\omega} + A \underline{u}_k + B^T \underline{\lambda}_k = \underline{f} \\ -D \dfrac{\underline{\lambda}_{k+1} - \underline{\lambda}_k}{\tau} + B \underline{u}_{k+1} - C \underline{\lambda}_k = \underline{g} \end{array}}$$

where the preconditioners $G = G^T > 0$ and $D = D^T > 0$ fulfil the spectral equivalence inequalities

$$(55) \left\{ \begin{array}{l} \underline{\gamma}_1 G \le A \le \overline{\gamma}_1 G, \\ \underline{\gamma}_2 D \le B G^{-1} B^T + C \le \overline{\gamma}_2 D, \end{array} \right. \qquad \underline{\gamma}_1 I \le G^{-0.5} A G^{-0.5} \le \overline{\gamma}_1 I$$

where $\underline{\gamma}_1, \overline{\gamma}_1, \underline{\gamma}_2, \overline{\gamma}_2 > 0$ "good" spectral equivalence const.

- With the substitution

(56) $\qquad \underline{\lambda}_k = D^{-0.5} \underline{\mu}_k$ and $\underline{u}_k = G^{-0.5} \underline{v}_k$

we see that (54) is equivalent to the classical Arrow-Hurwicz algorithm applied to the transformed system

$$(57) \left\{ \begin{array}{l} \dfrac{\underline{v}_{k+1} - \underline{v}_k}{\omega} + G^{-0.5} A G^{-0.5} \underline{v}_k + G^{-0.5} B^T D^{-0.5} \underline{\mu}_k = G^{-0.5} \underline{f} \\ -\dfrac{\underline{\mu}_{k+1} - \underline{\mu}_k}{\tau} + D^{-0.5} B G^{-0.5} \underline{v}_{k+1} - D^{-0.5} C D^{-0.5} \underline{\mu}_k = D^{-0.5} \underline{g} \end{array} \right.$$

- Convergence analysis and determination of the iteration parameters $\omega = \omega(\gamma_1, \gamma_2)$ and $\tau = \tau(\gamma_1, \gamma_2)$ see [ ] W. Queck: SIAM J. Num. Anal., v. 26, 1989, 1016-1030.