THEOREM 2.9: (The case $\bar{V}_{0h} \subset \bar{V}_0$)

Ass.: 1. Let the assumption of Theorem 2.7 be fulfilled.

2. Equilibrium condition: $\bar{V}_{0h} \subset \bar{V}_0$, i.e. $\alpha_1 = \tilde{\alpha}_1$.

St.: Then the following error estimate is valid:

$\exists!$ (2) $(2)_h$

$$(25) \quad \| \underset{\substack{\cap \\ V_g}}{u} - \underset{\substack{\cap \\ \bar{V}_{gh}}}{u_h} \|_X \leq 2\left(1+\frac{\alpha_2}{\alpha_1}\right)\left(1+\frac{\beta_2}{\tilde{\beta}_1}\right) \inf_{v_h \in X_h} \|u - v_h\|_X$$

Proof: $\|u-u_h\|_X \leq \|u-\tilde{v}_h\|_X + \|\tilde{v}_h - \tilde{u}_h\|_X$

- Let $\tilde{v}_h \in V_{gh}$. Then we have $\forall w_h \in \bar{V}_{0h} \subset \bar{V}_0$:

$$a(\underbrace{u_h - \tilde{v}_h}_{\in \bar{V}_{0h}}, w_h) = \underbrace{a(u_h, w_h) - a(u, w_h)}_{=0} + a(u-\tilde{v}_h, w_h)$$

$X \supset X_h$

Indeed:

$$\begin{array}{r|l} & a(u_h, w_h) + b(w_h, \lambda_h) = \langle f, w_h\rangle \ \forall w_h \subset \bar{V}_{0h} \\ - & a(u, w_h) + b(w_h, \lambda) = \langle f, w_h\rangle \ \forall w_h \subset \bar{V}_{0h} \\ \hline & a(u_h, w_h) - a(u, w_h) = \underbrace{b(w_h, \lambda - \lambda_h)}_{=0} \ \forall w_h \subset \bar{V}_{0h} \Leftarrow \hat{V}_0 \end{array}$$

$$= a(u-\tilde{v}_h, w_h) \leq \alpha_2 \|u-\tilde{v}_h\|_X \|w_h\|_X$$

- Setting $w_h = u_h - \tilde{v}_h \in \bar{V}_{0h}$, we get

$$\alpha_1 \|u_h - \tilde{v}_h\|_X^2 \leq a(u_h - \tilde{v}_h, u_h - \tilde{v}_h) \leq \alpha_2 \|u - \tilde{v}_h\|_X \|u_h - \tilde{v}_h\|_X$$

i.e. $\|u_h - \tilde{v}_h\|_X \leq \frac{\alpha_2}{\alpha_1} \|u - \tilde{v}_h\|_X$.

- Therefore, we have $\forall \tilde{v}_h \in \bar{V}_{gh}$

$$\|u-u_h\|_X \leq \|u-\tilde{v}_h\|_X + \|\tilde{v}_h - u_h\|_X \leq \left(1+\frac{\alpha_2}{\alpha_1}\right)\|u-\tilde{v}_h\|_X,$$

i.e.

$$\|u-u_h\|_X \leq \left(1+\frac{\alpha_2}{\alpha_1}\right) \inf_{\tilde{v}_h \in \bar{V}_{gh}} \|u-\tilde{v}_h\|_X \underset{\text{Lemma 2.8.}}{\leq} (\ ) \cdot 2\left(1+\frac{\beta_2}{\tilde{\beta}_1}\right) \inf_{z_h \in X_h} \|u - z_h\|_X$$

q.e.d.