LEMMA 2.8:

Ass.: ① The spaces X_h and Λ_h fulfill the discrete LBB-condition (18): $\tilde{\beta}_1$,

② $g \in \Lambda^*$,

③ $|b(v,\mu)| \leq \beta_2 \|v\|_X \|\mu\|_\Lambda \;\forall v \in X \;\forall \mu \in \Lambda$.

St.: Then, for any $u \in \bar{V}_g = \bar{V}(g)$, the following estimates are valid:

$$(22) \quad \inf_{z_h \in X_h} \|u - z_h\|_X \underset{\substack{\uparrow \\ \text{trivial, since} \\ X_h \supset \bar{V}_{gh}}}{\leq} \inf_{\tilde{v}_h \in \bar{V}_{gh}} \|u - \tilde{v}_h\|_X \leq 2\left(1 + \frac{\beta_2}{\tilde{\beta}_1}\right) \inf_{z_h \in X_h} \|u - z_h\|_X$$

Proof:

- Let $u \in \bar{V}_g$, i.e. $b(u,\mu) = \langle g, \mu \rangle \;\forall \mu \in \Lambda$, be given and let us consider the following constraint minimization problem (CMP)

$$(23) \quad \inf_{\tilde{v}_h \in \bar{V}_{gh}} \|u - \tilde{v}_h\|_X \underset{(\text{mms})}{\Longleftrightarrow} \inf_{\tilde{v}_h \in \bar{V}_{gh}} \left\{ \frac{1}{2}(\tilde{v}_h, \tilde{v}_h) - (u, \tilde{v}_h) \right\}$$

- The CMP (23) is equivalent to the MVP (24) (mms or see Section 2.2):

$$(24) \quad \begin{cases} (v_h, w_h) + b(w_h, \lambda_h) = (u, w_h) & \forall w_h \in X_h \\ b(v_h, \mu_h) = \langle g, \mu_h \rangle & \forall \mu_h \in \Lambda_h \end{cases}$$

- Then, for an arbitrary $z_h \in X_h$, we have

$$(v_h - z_h, w_h) + b(w_h, \lambda_h) = \overbrace{(u - z_h, w_h)}^{=: \langle F, w_h \rangle} \quad \forall w_h \in X_h$$

$$b(v_h - z_h, \mu_h) = \underbrace{b(u - z_h, \mu_h)}_{=: \langle g, \mu_h \rangle - b(z_h, \mu_h) =: \langle G, \mu_h \rangle} \quad \forall \mu_h \in \Lambda_h$$