Proof: see [GR], Lemma 8.2, pp. 406–407

1. $\exists!\ Pu \in U : \ J_U(Pu) = \inf_{v\in U} J_u(v)$ (21)

$$\Updownarrow \qquad \tfrac{1}{2}\|v\|^2 - (u,v) = \tfrac{1}{2}\|v-u\|^2 - \tfrac{1}{2}\|u\|^2$$

$$(Pu-u, v-Pu) \geq 0 \quad \forall v \in \bar{U} \qquad (22)$$

$$\Updownarrow$$

$$\tilde{J}_u(Pu) = \inf_{v\in\bar{U}} \tilde{J}_u(v) \equiv \inf_{v\in\bar{U}} \|v-u\|^2$$

$\exists$

- $U \neq \emptyset \wedge \tilde{J}_u(v) \geq 0 \ \forall v \in V \Rightarrow \inf_{v\in\bar{U}} \tilde{J}_u(v)$ is finite
- $\{\varepsilon_k\}$, $\varepsilon_k > 0$, $\varepsilon_k \searrow 0$. Then

  $\exists \{v_k\} \subset U : \ \tilde{J}_u(v_k) \leq \inf_{v\in\bar{U}} \tilde{J}_u(v) + \varepsilon_k$

  definition of "inf"

- $\|v_k - u\|^2 \leq \|z-u\|^2 + \varepsilon_k \quad \forall z \in \bar{U}$

  $\Rightarrow \|v_k\| \leq \|u\| + \|v^k - u\| \leq \|u\| + \sqrt{\|z-u\|^2 + \varepsilon_0}$

  $\Rightarrow \{v^k\} \subset U \subset V$ – * = bounded

$$\boxed{\text{weakly compact} \Leftarrow V^{**} = V \ \text{reflexive (H-space)}}$$

$\Rightarrow \exists\, v^* \in V : \ v_k \rightharpoonup v^*$ (wlg: $v_{k_i} \rightharpoonup v^*$)

$\Rightarrow v^* \in U$

$$\boxed{U = \bar{U} \text{ and convex} \Rightarrow U \text{ is weakly closed}}$$

$$\Rightarrow \tilde{J}_u(v^*) \leq \varliminf_{k\to\infty} \tilde{J}_u(v_k)$$

$$\boxed{\tilde{J}_u(\cdot) \text{ is continuous and convex} \Rightarrow \text{weakly lower semicont.}}$$

$$\Rightarrow \inf_{v\in\bar{U}} \tilde{J}_u(v) \leq \tilde{J}_u(v^*) \leq \varliminf_{k\to\infty} \tilde{J}_u(v_k) \leq \inf_{v\in\bar{U}} \tilde{J}_u(v) + \underbrace{\lim_{k\to\infty} \varepsilon_k}_{=0} \ \blacksquare$$