- The existence and uniqueness proof for the VI (19) is based on the equivalence of the VI (19) to a fixed point equation ($\Leftarrow$ Banach):

1. Define projection $P : V \mapsto \tilde{U}$: $u \mapsto Pu$

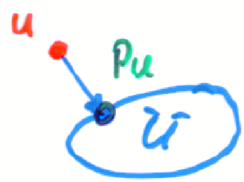
$$(21) \quad J_u(Pu) = \inf_{v \in \tilde{U}} J_u(v)$$

with $J_u(v) := \frac{1}{2}\|v\|^2 - (u,v) = \frac{1}{2}\|v-u\|^2 - \frac{1}{2}\|u\|^2$

2. The MP (21) is obviously equivalent to the VI (mms):

$$(22) \quad w \equiv Pu \in \tilde{U} : (Pu, v-Pu) \geq (u, v-Pu) \; \forall v \in \tilde{U} \; \forall u \in \tilde{U}.$$

3. The Minimum Problem



$$(23) \quad \text{Find } w \in \tilde{U} : \tilde{J}_u(w) = \inf_{v \in \tilde{U}} \tilde{J}_u(v)$$

with $\tilde{J}_u(v) := \frac{1}{2}\|v-u\|^2$ is equivalent to (21)$\equiv$(22).

4. The MP (23) has a unique solution $w = Pu \in \tilde{U}$ and the (nonlinear) operator $P : V \mapsto \tilde{U}$ is <u>non expansive</u>, i.e. $\|Pu - Pv\| \leq \|u-v\| \; \forall u,v \in \tilde{V}$.
<u>Proof</u>: see [GR], Lemma 8.2, pp. 406–407 ■

5. The VI (19) is equivalent to the nonlinear fixed point equation (mms)

$$(24) \quad \text{Find } u \in \tilde{V} : u = B_\varrho(u) := P[(I - \varrho J^{-1}A)u + \varrho J^{-1} f] \text{ in } \tilde{V}.$$

6. For sufficient small $\varrho \in (0, \varrho_0)$ the operator $B_\varrho : V \longmapsto V$ is <u>contractive</u> and, therefore, the fixed point equation (24) has a unique solution $u \in \tilde{U} \subset \tilde{V}$ that can be determined by the FPI

$$u_{n+1} = B_\varrho(u_n).$$