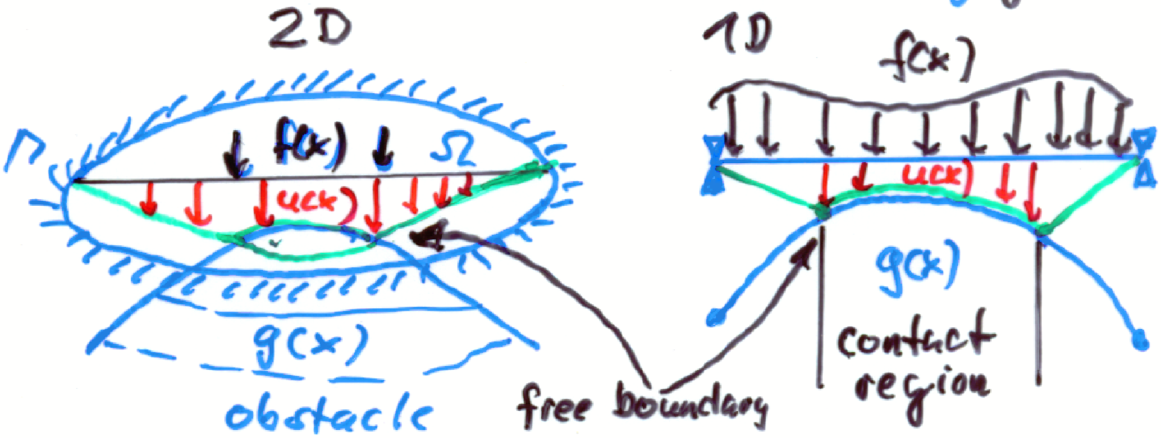
■ Example: Obstacle Problem

- **Problem Description:**

  Find the vertical displacement $u(\cdot)$ of a membran Ω which is loaded by vertical forces $f(\cdot)$ (force density), fixed on the boundary $\Gamma = \partial\Omega$ and located above an obstacle described by $g(\cdot)$:



- **Model:** Constraint Minimization Problem

(MP) Find $u \in \overline{U} := \{ v \in V = \overset{\circ}{H}{}^1(\Omega) : v(x) \geq g(x) \ \forall x \in \Omega \}$:

$$J(u) = \inf_{v \in \overline{U}} J(v),$$

with $J(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega f v \, dx$ and given

$f \in L_2(\Omega)$, $g \in \{ v \in H^1(\Omega) : v(x) \leq 0 \ \forall x \in \Gamma \}$.

- MP is obviously equivalent to the VI

(VI) Find $u \in \overline{U}$: $\int_\Omega \nabla^T u \, \nabla (v-u) \, dx \geq \int_\Omega f (v-u) \, dx \ \forall v \in \overline{U}$.

- **Exercise 1.9:** Show that a solution $u \in U \cap H^2(\Omega)$ of the MP $\equiv$ VI satisfies the following PDE ineq.:

$$\begin{cases} -\Delta u \geq f, \ u \geq g, \ (\Delta u + f)(u-g) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$