1.2.2. Variational Inequalities as Nonlinear Fixed Point Equations

- Let V (resp. V_0) be a Hilbert-space $(\|\cdot\|, (\cdot,\cdot))$ and $U \subset V$ a <u>subset</u> of V:

$$(18)\begin{cases} a) & U \neq \emptyset \quad \text{(non-empty)} \\ b) & U = \overline{U} \quad \text{(closed)} \\ c) & \lambda u + (1-\lambda) v \in U \ \forall \lambda \in [0,1] \ \forall u,v \in U \quad \text{(convex)} \end{cases}$$

Furthermore, let $f \in V^*$ and $a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}^1$ be a V-elliptic and V-bounded bilinear form.

- Then the <u>Variational Inequality</u> (VI)

$$(19)\quad \text{Find } u \in U : \underbrace{a(u,v-u)}_{\overset{!!}{=}\ \langle Au, v-u\rangle,\ A \in L(V,V^*)} \geq \langle f, v-u\rangle \ \forall v \in U$$

has a unique solution $u \in U$. If $U \subset V$ is a subspace of V, then (19) is obviously equivalent to (4).

- If the bilinear form $a(\cdot,\cdot)$ is additionally <u>symmetric</u>, then the variational inequality (19) is equivalent to the following constraint <u>minimization problem</u>:

$$(20)\quad \text{Find } u \in U : J(u) = \inf_{v \in U} J(v)$$

with the Ritz energy functional $J(v) = \frac{1}{2} a(v,v) - \langle f, v\rangle$.

<u>Proof:</u> mms with the hint

$$(20) \Longleftrightarrow J(u + t(v-u)) \geq J(u) \Longleftrightarrow (19)$$

$$\forall t \in [0,1] \ \forall v \in U$$

q.e.d.