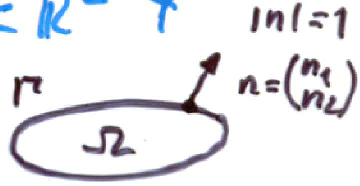
- **Example 1.3:** The biharmonic equation

Let us consider the biharmonic equation

$$\Delta^2 u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^2$$



under homogeneous
Dirichlet Boundary Conditions

$$u = 0 \quad \text{and} \quad \partial_n u := \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma = \partial\Omega.$$

Derivation of the VF: ① – ⑤

① $V_0 = \{ v \in V = H^2(\Omega) : u = \partial_n u = 0 \text{ on } \Gamma \} = \overset{0}{H}{}^1(\Omega)$

② $\int\limits_\Omega \Delta^2 u \cdot v \, dx = \int\limits_\Omega f v \, dx$

③ $\int\limits_\Omega \Delta \Delta u \cdot v \, dx = \int\limits_\Omega \operatorname{div} \nabla(\Delta u) \cdot v \, dx$

$$= - \int\limits_\Omega \nabla(\Delta u) \cdot \nabla v \, dx + \int\limits_\Gamma \partial_n \Delta u \cdot v \, ds$$

$$\int\limits_\Omega \partial_i u \cdot v \, dx = - \int\limits_\Omega u \cdot \partial_i v \, dx + \int\limits_\Gamma u \cdot v \cdot n_i \, ds_x$$

$w = uv$

$$\int\limits_\Omega \partial_i w \, dx = \int\limits_\Gamma w \cdot n_i \, ds$$

$$= + \int\limits_\Omega \Delta u \cdot \Delta v \, dx - \int\limits_\Gamma \Delta u \cdot \underbrace{\partial_n v}_{=0} \, ds + \int\limits_\Gamma \partial_n \Delta u \cdot \underbrace{v}_{=0} \, ds$$

④ Include the natural BC (here there are no nat. BC!)

⑤ $V_g := \{ v \in V = H^2(\Omega) : v = g_0 := 0, \ \partial_n v = g_1 := 0 \} = V_0$

Result: VF of the 1st biharmonic BVP

Find $u \in V_0 = \overset{0}{H}{}^2(\Omega) : a(u,v) = \langle f, v \rangle \ \forall v \in V_0$,

where $a(u,v) := \int\limits_\Omega \Delta u \, \Delta v \, dx$

$\langle f, v \rangle := \int\limits_\Omega f \cdot v \, dx$, $f \in L_2(\Omega)$ – given

$\Gamma = \partial\Omega \in C^{0,1}$

Tutorial 2: $\exists!$ ....