SUMMARY: FVM and Discrete Convergence:

$-\operatorname{div}(a\nabla u) + cu = f$ in Ω, $u = 0$ on $\Gamma = \partial\Omega$

$u: \bar{\Omega} \to \mathbb{R}$

$$A \rightsquigarrow Lu(x) = -\frac{1}{|B(x)|} \sum_{E \in \partial B(x)}' \int_E a(y) \frac{\partial u}{\partial n}(y)\, ds_y + \frac{1}{|B(x)|} \int_{B(x)} c(y) u(y)\, dy$$

$V = u_h: \bar{\omega}_h \to \mathbb{R}$

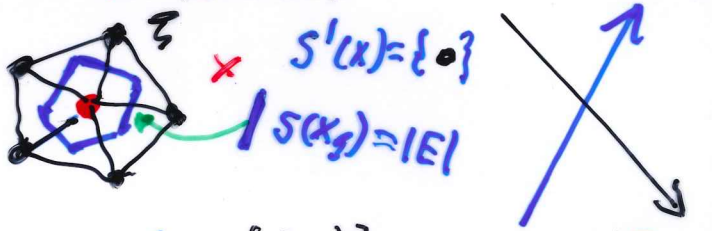
$$A_h \rightsquigarrow L_h V(x) = -\frac{1}{|B(x)|} \sum_{\xi \in S(x)}' \bar{a}(x_\xi) \frac{V(\xi) - V(x)}{|\xi - x|} s(x_\xi) + \bar{c}(x) V(x), \quad x \in \omega = \mathring{\omega}, \; V(x) = 0, \; \forall x \in \delta\omega$$

$\omega_h = \mathring{\omega}_h$

$$D(L) := \mathring{H}^1(\Omega) \cap H^2(\Omega) \subset X = \mathring{H}^1(\Omega) \xrightarrow{L} Y = H^{-1}(\Omega) : Lu = f := \frac{1}{|B(x)|} \int_{B(x)} f(y)\, dy : u \leftarrow E_h u_h$$

Restriction to the grid $\bar{\omega}$

$R_h \downarrow$ $\uparrow E_h$ $\qquad$ $Q_h \downarrow$ $\uparrow F_h$



$$X_h = \mathring{H}^1(\omega) \xrightarrow{L_h} Y_h = H^{-1}(\omega) : L_h u_h = f_h := [f(x)]_{x \in \omega} : u_h \to R_h u$$

Discrete Error: $z_h = R_h u - u_h \in X_h = \mathring{H}^1(\omega)$: $L_h z_h = \psi_h \Rightarrow z_h = L_h^{-1} \psi_h$

$$\Rightarrow \|z_h\|_{X_h} \le \underbrace{\|L_h^{-1}\|_{L(Y_h, X_h)}}_{\le c_S} \cdot \underbrace{\|\psi_h(u)\|_{Y_h}}_{\le c_A(u) h^p} \le c_S c_A(u) h^p \qquad (p = 1 \text{ if } u \in H^2)$$

Stability + Approximation $\Rightarrow$ discrete Convergence

with the appr. error $\psi_h(u) := L_h R_h u - Q_h L u = -\frac{1}{|B(x)|} \Big[\sum_{\xi \in S'(x)} a(x_\xi) \frac{u(\xi) - u(x)}{|\xi - x|} s(x_\xi) - \sum_{E \in \partial B(x)}' \int_E a(y) \frac{\partial u}{\partial n}(y)\, ds_y \Big] + \ldots$