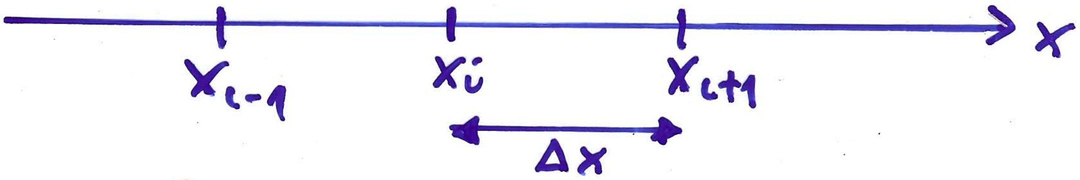
4.2. FDM and FVM

FDM = Finite Difference Methods:

- **Classical Idea:**
  Replace partial derivatives by finite differences on a uniform grid:

$$u_{\bar{x}}(x_i,y) = \frac{u(x_i,y)-u(x_{i-1},y)}{\Delta x} \approx \frac{\partial u}{\partial x}(x_i,y) \approx \frac{u(x_{i+1},y)-u(x_i,y)}{\Delta x} =: u_x(x_i,y)$$

backward diff. $O(h)$ $O(h)$ forward diff



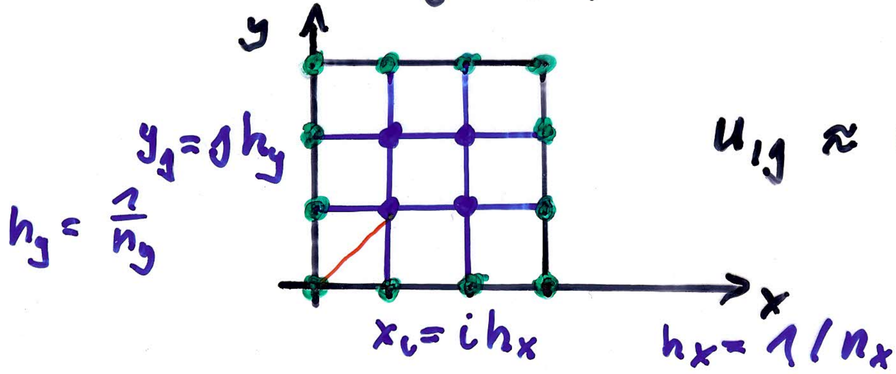
$O(h^2)$ Taylor

$$\frac{\partial^2 u}{\partial x^2}(x_i,y) \approx \frac{u(x_{i+1},y)-2u(x_i,y)+u(x_{i-1},y)}{(\Delta x)^2}$$

$=: u_{\bar{x}x}(x_i,y)$ - 2nd diff.

- **Poisson Equation:**

$$(1)\quad \begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega = (0,1)^2 \\ u(x,y) = g(x,y), & (x,y) \in \Gamma = \partial\Omega \end{cases}$$



$u_{ij} \approx u(x_i,y_j)$

$h_x = h_y = h$
$n_x = n_y = n$

$$(1)_h \begin{cases} -\dfrac{u_{i-1,j}-2u_{ij}+u_{i+1,j}}{h_x^2} - \dfrac{u_{i,j-1}-2u_{ij}+u_{i,j+1}}{h_y^2} = f_{ij} := f(x_i,y_j) & i,j=\overline{1,n-1} \\ u_{ij} = g(x_i,y_j) \text{ for } (x_i,y_j)\in\Gamma \end{cases}$$

Find $\underline{u}_h = [u_{ij}]_{i,j=\overline{1,n-1}}$: $K_h \underline{u}_h = \underline{f}_h$ in $\mathbb{R}^{N_h}$, $N_h = (n-1)^2$

$$K^{FDM} = \frac{1}{h^2} K^{FEM}$$