4.1.3. Convergence in the DG-norm $\|\cdot\|_h$

Lemma 4.9:

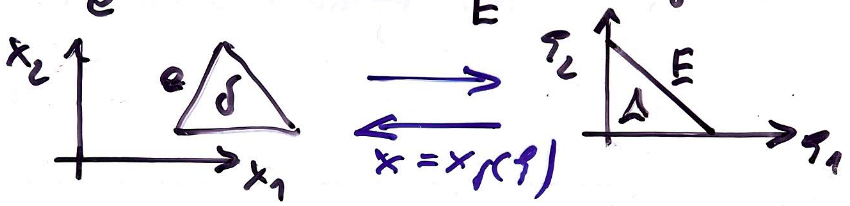
There is some generic constant $c = \text{const} > 0$:

$$(9) \quad \|v\|_e \le c \underbrace{\left(\frac{|e|}{|\delta|}\right)^{1/2}}_{\simeq h_\delta^{-1/2}} \left(\|v\|_\delta + h_\delta \|\nabla v\|_\delta\right)$$

$$\forall v \in H^1(\delta) \ \forall e \in \partial\delta \ \forall \delta \in T_h, \ \forall h \in Ⓗ.$$

Proof:

$$\|v\|_e^2 = \int_e (v(x))^2 \, ds_x = \int_E (\overbrace{v(x_\delta(\xi))}^{\tilde v(\xi)})^2 |e| \, ds_\xi$$



$$= |e| \int_E (\tilde v(\xi))^2 \, ds_\xi$$

trace theorem

$$\le c\,|e| \left[\int_\Delta v^2(x_\delta(\xi))\, d\xi + \int_\Delta |\underbrace{\nabla_\xi}_{= J_\delta \nabla_x} v(x_\delta(\xi))|^2 \, d\xi \right]$$

$\Delta \to \delta$

$$\le c\,|e| \left[\int_\delta v^2(x) \frac{1}{|\delta|}\, dx + \int_\delta \|J_\delta\|^2 |\nabla_x v|^2 \frac{1}{|\delta|}\, dx \right]$$

$$\le c \frac{|e|}{|\delta|} \left[\|v\|_\delta^2 + h_\delta^2 \|\nabla v\|_\delta^2 \right]$$

$\sqrt{}$ + $\sqrt{a^2+b^2} \le a+b, \ \forall a,b \ge 0 \Rightarrow (9)$ q.e.d.

Remark:

$$(9)_\varepsilon \quad \|v\|_e \le c\, h_\delta^{-1/2} \left(\|v\|_\delta + h_\delta^{1/2+\varepsilon} |v|_{H^{1/2+\varepsilon}(\delta)} \right)$$

$$\forall v \in H^{1/2+\varepsilon}(\delta) \ \forall e \in \partial\delta \ \forall \delta \in T_h \ \forall h \in Ⓗ$$

Proof: Use the trace theorem

$$\|\tilde v\|_E = \|\tilde v\|_{L_2(E)} \le c\, \|\tilde v\|_{H^{1/2+\varepsilon}(\Delta)}$$