- Let us define the DG norm

(6) $\|v\|_h^2 := \sum_{\delta\in T_h} \|\nabla u\|^2_{L_2(\delta)} + \sum_e \frac{\alpha_e}{h_e}\|[v]_e\|^2_{L_2(e)}$

that is indeed a norm on $\bar{V}_k(T_h)$! (mms)
Now we are in the position to prove $\bar{V}_k(T_h)$-ellipticity and $\bar{V}_k(T_h)$-boundedness of the DG bilinear form $a_h(\cdot,\cdot)$:

- <u>Lemma 4.5:</u> ($\bar{V}_k(T_h)$-ellipticity)
Let $\beta = -1$ (SIPG). Then there exists a positive constant $\mu_3 \neq \mu_3(h)$:
(7) $a_h(v,v) \geq \mu_3 \|v\|_h^2 \quad \forall v \in \bar{V}_k(T_h)$
provided that all α_e are sufficiently large, i.e. $\exists \alpha_0 = \text{const} > 0: \forall \alpha_e \geq \alpha_0$, (7) holds.

<u>Proof:</u> $\forall v \in \bar{V}_k(T_h)$, $\|\cdot\|_\delta := \|\cdot\|_{L_2(\delta)}, \ldots$

$$a_h(v,v) = \sum_\delta \|\nabla v\|_\delta^2 - 2\underbrace{\sum_e (\{\nabla v\},[v])_e}_{(*)} + \sum_e \frac{\alpha_e}{h_e}\|[v]\|_e^2$$

$$(*) \leq \Big(\tilde{c}\sum_\delta \|\nabla v\|_\delta^2\Big)^{1/2}\Big(\sum_e \frac{1}{\alpha_e}\frac{\alpha_e}{h_e}\|[v]_e\|_e^2\Big)^{1/2}$$

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$$

$$\geq \underbrace{[1-\tilde{c}\varepsilon]}_{\tilde{c}\varepsilon = \frac{1}{2}} \sum_\delta \|\nabla v\|_\delta^2 + \sum_e \Big[1 - \frac{1}{\alpha_e\varepsilon}\Big]\frac{\alpha_e}{h_e}\|[v]\|_e^2$$

$\varepsilon = 1/(2\tilde{c})$ $\qquad 1 - \frac{2\tilde{c}}{\alpha_e} \geq \frac{1}{2}$ $\qquad \alpha_e \geq 4\tilde{c}$ L 4.4.

$$= \frac{1}{2}\sum_\delta \|\nabla v\|_\delta^2 + \frac{1}{2}\sum_e \frac{\alpha_e}{h_e}\|[v]\|_e^2$$

$$= \frac{1}{2}\|v\|_h^2,$$ i.e. $\mu_3 = \frac{1}{2}$ provided that $\alpha_e \geq 4\tilde{c}$.

q.e.d.