4. Other Discretization Methods

4.1. Discontinuous Galerkin (DG) Methods = Non-conforming FEM

4.1.1. Derivation of the DG Variational Formulations and the DG Schemes

- Let us again consider the <u>model problem</u>

(1) $\boxed{\begin{array}{l} \text{Given } f \in L_2(\Omega), \text{ find } u \in V = H^1(\Omega) \text{ such that} \\ -\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^d \quad (\Gamma \wedge \mathrm{Lip},\ d = 2,3) \\ u = g \quad \text{on } \Gamma = \partial\Omega \end{array}}$

- <u>Solvability:</u>
  Due to Lax-Milgram, there exists a unique weak solution $u \in V_0 = \overset{\circ}{H}{}^1(\Omega) \subset V = H^1(\Omega)$:

$$(1)_{VF} \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in \overset{\circ}{H}{}^1(\Omega)$$

$$\overset{\shortparallel}{a(u,v)} = (f,v)_{0,\Omega} =: \langle F, v \rangle \quad \forall v \in V_0$$

- <u>Exercise 4.1:</u> (see also Chapter 2)
  Show that $\nabla u \in H(\mathrm{div},\Omega)$ and $\mathrm{div}\, \nabla u = -f \in L_2(\Omega)$!
  Therefore, $u \in \overset{\circ}{H}{}^1(\Omega) \cap H^\Delta(\Omega)$, where

$$H^\Delta(\Omega) := \{ u \in H^1(\Omega) : \Delta u = \mathrm{div}\, \nabla u \in L_2(\Omega) \}$$

$$= \{ u \in H^1(\Omega) : \nabla u \in H(\mathrm{div},\Omega) \}.$$