4.4. Boundary Integral Operators (BIO) and their Properties

4.4.1. Formal Definition of the most important BIO

- Let $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ be a simply connected, bounded domain with a sufficiently smooth boundary $\Gamma = \partial\Omega$.
  Furthermore, let $u(\cdot)$ be some harmonic function in Ω, i.e.

(7) $-\Delta_x u(x) = 0 \quad (L_x u(x) = 0)$ in $\Omega =$ .

Then the normal derivative of the representation formula (3), cf. Subsect. 4.2.3,

(8) a) $$\underbrace{\tfrac{1}{2}}_{\sigma(y)} u(y) = \underbrace{-\int_\Gamma u(x) \frac{\partial E}{\partial n_x}(x,y)\, ds_x}_{\text{double layer potential DLP}} + \underbrace{\int_\Gamma v(x) E(x,y)\, ds_x}_{\text{single layer potential SLP}}, \quad y \in \Gamma$$

gives us the relation

(8) b) $$\frac{1}{2} \underbrace{\frac{\partial u}{\partial n_y}(y)}_{=v(y)} = \underbrace{-\frac{\partial}{\partial n_y} \int_\Gamma u(x) \frac{\partial E}{\partial n_x}(x,y)\, ds_x}_{\text{hypersingular BIO}} + \underbrace{\int_\Gamma v(x) \frac{\partial E}{\partial n_y}(x,y)\, ds_x}_{\text{adjoint DLP}}, \quad y \in \Gamma$$

with the Neumann data

$$v = \frac{\partial u}{\partial n_x}\Big|_\Gamma = (\nabla_x u(x), n_x(x))\big|_\Gamma$$

and the fundamental solution of $(-\Delta)$

(9) $$E(x,y) = \begin{cases} -\frac{1}{2\pi} \ln|x-y| & \text{in 2D } (d=2) \\ \frac{1}{4\pi} \frac{1}{|x-y|} & \text{in 3D } (d=3) \end{cases}$$