4.2. Formulation of BVPs as Integral Equations

4.2.1. Interior and Exterior BVPs for the Laplace Oper.

- Let $\Omega \subset \mathbb{R}^d$ be *, simply connected domain with the boundary $\Gamma = \partial\Omega \in C^{0,1}$ resp. $C^{0,1} \cap PC^k$ resp. C^k (sufficiently smooth!), $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$, $d = 2,3$.

- Consider the <u>Poisson-Equation</u>

$(1)_i$ $-\Delta u(x) = f(x)$ in Ω + BC on Γ (interior BVP),

$\underset{\text{Laplace}}{f \equiv 0}$

$(1)_e$ $-\Delta u(x) = f(x)$ in Ω^c + BC on Γ + SC for $|x| \to \infty$ (ext. BVP).

- <u>Boundary Conditions (BC):</u>

I. $u(x) = g_D(x)$, $x \in \Gamma_D = \Gamma$ (= 1st BVP = Dirichlet BVP),

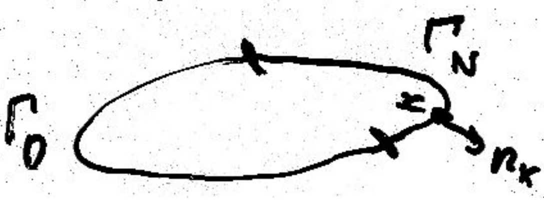
II. $\frac{\partial u}{\partial n_x}(x) = (\nabla u, n_x) = g_N(x)$, $x \in \Gamma_N = \Gamma$ (= 2nd BVP = N.)

Solution condition for the Neumann problem:

$$\int_\Omega f(x)\,dx + \int_\Gamma g_N(x)\,ds_x =: \langle F, 1 \rangle = 0\,!$$

III. $\frac{\partial u}{\partial n_x}(x) + \alpha(x)(u(x) - g_R(x)) = 0$, $x \in \Gamma_R = \Gamma$ (= 3rd BVP)

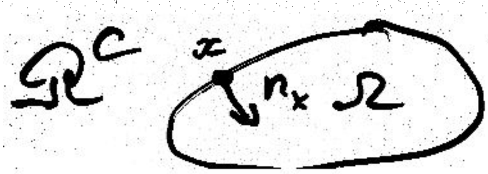
IV. Mixed BVP:



$u(x) = g_D(x)$ $\forall x \in \Gamma_D$

$\frac{\partial u}{\partial n_x}(x) = g_N(x)$ $\forall x \in \Gamma_N$

- <u>Sommerfeld's decay condition for $|x| \to \infty$:</u>



$d = 3$

$|u(x)| = O\left(\frac{1}{|x|}\right)$ for $|x| \to \infty$.