b) Approximation Estimate: $\|\psi\|_{W_2^{-1}(\omega)} \leq$?
by mapping to a reference domain,
application of Bramble-Hilbert's Lemma,
and return mapping: Under the assumption

- (i) $u \in W_2^2(\Omega)$,
- $\bar{\omega}$ - regular grid, i.e. T_Δ - regular triang.,
- and additional smoothness requirements imposed on the data $\{a, c, f\}$,

we can proof the estimate

$$(28) \quad |(\psi, z)| \leq c(u)\, h\, \|z\|_{W_2^1(\omega)}$$

Indeed, from the splitting of the approximation error

$x \in \omega$

$$\psi(x) = \overbrace{L_h u}^{A_h R_h u} - \underbrace{L_h v}_{=\bar{f}\ (3)} =$$

$$= L_h u - \underbrace{\left[-\frac{1}{H(x)} \int_{\partial\mathcal{H}(x)} a(y) \frac{\partial u}{\partial n}(y)\, ds_y + \frac{1}{H(x)} \int_{\mathcal{H}(x)} c u\, dy\right] + \frac{1}{H(x)} \int_{\mathcal{H}(x)} f dy - \bar{f}(x)}_{= Lu - f \;=\; 0}$$

$$= \left\{-\frac{1}{H(x)} \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \frac{u(\xi) - u(x)}{h(x,\xi)} s(x_\xi) - \left[-\frac{1}{H(x)} \int_{\partial\mathcal{H}(x)} a(y) \frac{\partial u}{\partial n}(y)\, ds_y\right]\right\}$$

$$+ \left\{\bar{c}(x) u(x) - \frac{1}{H(x)} \int_{\mathcal{H}(x)} c u\, dy\right\} + \underbrace{\left\{\frac{1}{H(x)} \int_{\mathcal{H}(x)} f(y)\, dy - \bar{f}(x)\right\}}_{= 0 \text{ for } \bar{f}(x) := \frac{1}{H(x)} \int_{\mathcal{H}(x)} f(y) dy}$$

$$= \psi_a(x) + \overset{O(h)}{\psi_{cu}(x)} + \underset{=0}{\overset{O(h)}{\psi_f(x)}} = \psi_a(x) + \psi_{cu}(x)$$

$$\Rightarrow |(\psi, z)| \leq \underset{1)}{|(\psi_a, z)|} + \underset{2)\ \text{OK}}{|(\psi_{cu}, z)|}$$