3.4.2. Discrete Convergence of the classical Integral Balance Method

- For the DS $(6)_{PB}$ (similarly, for $(6)_{MD}, \dots$)

$$
(6) \quad \boxed{\begin{array}{lll} V = \underline{u}_h : \overline{\omega}_h \to \mathbb{R}: & L_h v(x) = f_h(x), \; x \in \omega & (6)_L \\ & \ell_h v(x) = g_h(x), \; x \in \gamma_h = \gamma_{2,3} & (6)_\ell \\ & v(x) = g_1(x), \; x \in \gamma_e = \gamma_1 & \end{array}}
$$

error estimates in discrete norms follow from the STABILITY and the APPROXIMATION with respect to the corresponding norms:

① Discrete Convergence in the $\overset{\circ}{W}{}_2^1(\omega_h)$-norm!

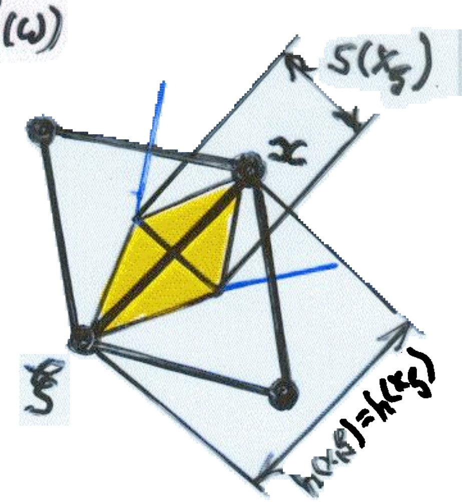
$$
(24) \quad \|\underset{\substack{(3)\\ \| \\ (1)}}{\overset{\text{error}}{u}} - \underset{(6)}{v}\|_{W_2^1(\omega_h)} \le c(u) \begin{cases} h, & \text{for (i) } u \in W_2^2(\Omega), \; \omega \text{ regular} \\ h^{3/2}, & \text{for (ii) } u \in W_2^3(\Omega), \text{ loc. non-u.} \\ h^2, & \text{for (iii) } u \in W_2^3(\Omega), \text{ unif., } \Gamma_1 = \Gamma \\ & \text{and data suff. smooth!} \end{cases}
$$

where $\overset{\circ}{W}{}_2^1(\omega_h) = \overset{\circ}{H}{}^1(\omega) := \{ z : \overline{\omega} \to \mathbb{R}^1 : z|_{\gamma_1} = 0 \}$ with the norm

$$
\|z\|^2_{W_2^1(\omega)} := \underbrace{\sum_{x_\xi} z^2_{\vec{n}}(x_\xi) \, H'(x_\xi)}_{=: |z|^2_{W_2^1(\omega)}} + \sum_{x \in \omega} z^2(x) H(x) + \sum_{x \in \gamma_n} z^2(x) h(x)
$$

with $z_{\vec{n}}(x_\xi) := \dfrac{z(\xi) - z(x)}{h(x, \xi)}$

$$
H'(x) = s(x_\xi) \, h(x, \xi)
$$



(iii) uniform grid: 

(ii) locally non-uniform grids: uniformity is only perturbed for $O(h^{-1})$ triangles (along the boundary, interfaces)