3.3.2. Construction via GALERKIN-PETROV Variational Technique

- For the sake of simplicity, we consider the homogeneous Dirichlet problem:

$$(7)_{SF} \quad \boxed{\begin{array}{l} -\operatorname{div}(a(x)\nabla u(x)) + c(x)u(x) = f(x),\ x\in\Omega, \\ u(x) = 0,\ x \in \Gamma_1 = \Gamma_3 = \Gamma_2 = \Gamma := \partial\Omega \end{array}}$$

in a polygonal, bounded (*) domain $\Omega \subset \mathbb{R}^{d=2}$ under the assumptions

$$(8) \begin{cases} a(x) \in (P)\, C^1(\overline{\Omega}):\ 0 < \overline{\mu}_1 \le a(x) \le \overline{\mu}_2 \quad \forall x\in\overline{\Omega}, \\ c(x) \in (P)\, C(\overline{\Omega}):\ 0 \le c(x) \le \overline{c} \quad \forall x \in \overline{\Omega}, \\ f \in L_2(\Omega) \\ \Gamma = \partial\Omega \in C^{0,1} \cap PC^k, \text{ with some } k \ge 2. \end{cases}$$

- Variational Formulation:

$$(7)_{VF} \quad \boxed{\text{Find } u \in V_0 = \overset{\circ}{W}{}^1_2(\Omega):\ a(u,v) = \langle F, v\rangle \ \forall v \in V_0}$$

with $a(u,v) = \int_\Omega (a(x)\nabla^T u \nabla v + c(x)u(x)v(x))\,dx$,

$\langle F, v\rangle = \int_\Omega f(x)v(x)\,dx$.

Assumptions (8) immediately yield

$$\left.\begin{array}{l} a(\cdot,\cdot) \text{ is } V_0\text{-elliptic and } V_0\text{-}{*} \\ F \in V_0^* = W_2^{-1}(\Omega) \end{array}\right\} \xRightarrow{} \exists!\, u \in V_0 : (7)_{VF}$$

Lax-Milgram