**Remark 3.4:** Approximation of the Convection

The convection term

$$\int_{\mathcal{H}(x)} (b, \nabla u)\, dy$$

can be approximated in such a way (upwind-approx.) that the monotony (1st) of $L_h$ ($\uparrow$ and, therefore, the diskrete maximum principle!) is preserved.

Starting point for deriving such approximations are the following relations:

for $x \in \omega = \mathring{\omega} \cup \gamma_N$ :

$$\int_{\mathcal{H}(x)} (b, \nabla u)\, dy = \int_{\partial \mathcal{H}(x)} (b, \vec{n}) \cdot u \, ds - \int_{\mathcal{H}(x)} \operatorname{div} b \cdot u \, dy$$

$$\underset{u=1}{\overset{\uparrow}{=}} \int_{\partial \mathcal{H}(x)} (b(y), \vec{n}(y)) [u(y) - u(x)]\, ds_y + \int_{\mathcal{H}(x)} \operatorname{div} b(y) [u(x) - u(y)]\, dy$$

$$\left[ \int_{\mathcal{H}(x)} \operatorname{div} b(y)\, dy - \int_{\partial \mathcal{H}(x)} (b, \vec{n})\, ds \right] u(x) = 0$$

$$= \quad I(x) \quad + \quad g(x)$$

$$\underset{\operatorname{div} b = 0}{\overset{\uparrow}{=}} \quad I(x) = \int_{\partial \mathcal{H}(x)} (b(y), \vec{n}(y)) [u(y) - u(x)]\, ds_y \approx \ldots$$

(B. Heinrich)

**1D-Example:**

$u : [0,1] \longrightarrow \mathbb{R}^1$

$-u''(x) + b\, u'(x) = 0$

$x \in (0,1)$

$u(0) = 0$

$u(1) = 1$

$v = u_h : \overline{\omega}_h = \{ x_i = ih : i = \overline{0,n} \} \to \mathbb{R}^1$:

$$-v_{\bar{x}x,i} + b \begin{Bmatrix} v_{x,i}, & \text{if } b < 0 \\ v_{\bar{x},i}, & \text{if } b > 0 \end{Bmatrix} = 0, \quad i = \overline{1,n}$$

$v_0 = 0$ $\qquad v_i \approx u(x_i)$

$v_n = 1$

with $v_{x,i} = \dfrac{v_{i+1} - v_i}{h}$, $v_{\bar{x},i} = \dfrac{v_i - v_{i-1}}{h}$, $v_{\bar{x}x,i} = \dfrac{v_{i+1} - 2v_i + v_{i-1}}{h^2}$.