DWR (Dual Weighted Residuals) Method

Goal-orientated a-posteriori error estimators:

In practice, instead of the solution $u \in V_g$ (or $u \in V_0$ after homogenization), we are often interested in the value $\ell(u) \in \mathbb{R}$ of a linear functional $\ell \in V_0^*$ on the solution $u \in V_0$, i.e. we need an a-posteriori error estimator of the error:

$$(46) \qquad |\ell(u) - \ell(u_h)| \leq \; ?$$

<u>Examples:</u>

1) $\ell(v) := \dfrac{(u-u_h, v)_0}{\|u-u_h\|_0}$: $\ell(u)-\ell(u_h) = \|u-u_h\|_0$

2) $\ell(v) := \int_{\Omega_0} v(x)\,dx$: average of v over $\Omega_0 \subset \Omega$

3) $\ell(v) := \int_{\Omega} \delta_\varepsilon(x-y)\, v(y)\,dx$: $\ell(u)-\ell(u_h) \approx u(y) - u_h(y)$

4) $\ell(v) := \int_{\Gamma_0} \frac{\partial v}{\partial N}\,ds$: Flux of v through $\Gamma_0 \subset \Gamma = \partial\Omega$?

etc.

To derive a-posteriori error estimates for (46), similar to the apriori L_2-estimate, we use a duality argument, i.e. we consider the auxiliary adjoint variational problem:

$$(47) \quad \text{Find } w \in V_0 : \; a(v, w) = \ell(v) \quad \forall v \in V_0$$

Now, take $v = u - u_h \in V_0$:

$$\ell(u) - \ell(u_h) = \ell(u-u_h) = a(u-u_h, w)$$

$$(\uparrow) \qquad = a(u-u_h, w-w_h) \quad \forall w_h \in V_{0h}$$

$$= \langle F, w-w_h \rangle - a(u_h, w-w_h)$$

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