2.7. Sobolev's Embedding Theorems

Def. 2.23:

X, Y - B-spaces: $X \subset Y$ (as sets)
The embedding operator $E: X \longrightarrow Y$ assigns every element $u \in X$ the same element $u \in Y$ (after identification!).
The embedding is called continuous resp. compact iff the embedding operator E is continuous (*) resp. compact.
Notation: $X \subset Y$ - continuous embedding,
$X \hookrightarrow Y$ - compact embedding.

Theorem 2.24: (Embedding for $\overset{\circ}{W}{}_p^1(\Omega)$)

Ass.: $\Omega \subset \mathbb{R}^d$ * ($\partial\Omega \in C^{0,1}$ is not required!)
$1 < p, q < \infty$

St.: 1. $\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow C(\overline{\Omega})$ if $p > d$,
i.e., for every function (equivalence class of functions) $u \in \overset{\circ}{W}{}_p^1(\Omega)$, there exists an equivalent function $u \in C(\overline{\Omega})$, and the embedding operator $E \in L(\overset{\circ}{W}{}_p^1(\Omega), C(\overline{\Omega}))$ and compact.

2. Let now $p \le d$. Then
a) $\overset{\circ}{W}{}_p^1(\Omega) \subset L_q(\Omega)$ if $q \le q_* := \frac{pd}{d-p} \wedge q < \infty$,
b) $\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow L_q(\Omega)$ if $q < q_*$.

3. Let $p \le d$ and $\Omega_m = \Omega \cap \mathcal{H}_m$: $\operatorname{meas}_{\mathbb{R}^m}(\Omega_m) > 0$, where $\mathcal{H}_m$ is a m-dim. hyperplan: $d-p \le m$, $m \le d$, $m \ge 1$.
a) $\overset{\circ}{W}{}_p^1(\Omega) \subset L_q(\Omega_m)$ if $q \le q_* := \frac{pm}{d-p} \wedge q < \infty$,
b) $\overset{\circ}{W}{}_p^1(\Omega) \hookrightarrow L_q(\Omega_m)$ if $q < q_*$.