$H^{-1/2}(\Gamma) = \gamma_n H(\operatorname{div})$

- **Theorem 2.20:** ($H(\operatorname{div})$ inverse trace theorem)
  Let $q_n \in H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))^*$. Then there exists a $q \in H(\operatorname{div})$ such that

  (19) $\gamma_n q = q_n$ and $\|q\|_{H(\operatorname{div})} \le c_e \|q_n\|_{H^{-1/2}(\Gamma)}$.

  If q_n satisfies $\langle q_n, 1 \rangle = 0$, then there exits an extension $q \in H(\operatorname{div})$ such that $\operatorname{div} q = 0$.

  Proof: Consider the weak solution of the Neumann problem

  $$(20) \quad \begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q_n & \text{on } \Gamma. \end{cases}$$

  Since $q_n \in H^{-1/2}(\Gamma)$, we get from Lax-Milgram that $\exists!\, u \in H^1(\Omega)$: (mms)

  $$(21) \quad \|\nabla u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \le c\, \|q_n\|^2_{H^{-1/2}(\Gamma)}.$$

  Now, setting $q = \nabla u$, we observe
  1) $\operatorname{div} q = \operatorname{div} \nabla u = \Delta u \overset{(20)}{=} u \in L_2(\Omega)$ (weak!)
  2) $\|q\|_0^2 + \|\operatorname{div} q\|_0^2 = \|\nabla u\|_0^2 + \|u\|_0^2 \overset{(21)}{\le} c\, \|q_n\|^2_{H^{-1/2}(\Gamma)}$.

  If $q_n : \langle q_n, 1 \rangle = 0$, then consider

  $$(22) \quad \begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q_n & \text{on } \Gamma \end{cases} \quad \wedge \ \langle q_n, 1 \rangle = 0.$$

  $\exists (!)$ L&M

  Again, set $q = \nabla u$ yielding $\operatorname{div} q = \operatorname{div} \nabla u = \Delta u \overset{(22)}{=} 0$.

  q.e.d.

- **Exercise 2.21:** Let $\Omega_1, \dots, \Omega_m$ be a non-overlapping domain decomposition of $\overline{\Omega} = \cup \overline{\Omega}_i$, with $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$.
  Let $q_i \in H(\operatorname{div}, \Omega_i)$: $\gamma_{n_i, \Gamma_{ij}} q_i = \gamma_{n_i, \Gamma_{ij}} q_j$
  $\forall \Gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$, $i, j = \overline{1, m}$.
  Then $q := \{ q = q_i \text{ on } \Omega_i,\ i = \overline{1,m} \} \in H(\operatorname{div})$.