■ IbyP-Formula and Traces of H(div) Functions:

• <u>Starting Point:</u> $\forall q=(q_1,\dots,q_d)^T \in [C^1(\bar\Omega)]^d$, $v \in C^1(\bar\Omega)$:

$$\int_\Omega \operatorname{div} q \cdot v \, dx = -\int_\Omega q \cdot \nabla v \, dx + \int_\Gamma q \cdot n \cdot v \, ds \text{, i.e.}$$

$$(17)\quad \int_\Gamma q\cdot n \cdot v \, ds = \int_\Omega (\operatorname{div} q \cdot v + q \cdot \nabla v) dx \quad \forall q \in C^1(\bar\Omega)^d, \forall v \in C^1(\bar\Omega)$$

$$\overset{!!}{=} \langle \gamma_n q, \gamma_0 v \rangle_{H^{-1/2}\times H^{1/2}}, \quad \gamma_n q \in (H^{1/2}(\Gamma))^*, \ \gamma_0 v \in H^{1/2}(\Gamma)$$

closure principle ↓ $q \in H(\operatorname{div})$, $v \in H^1(\Omega)$

$$H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))^*$$

• <u>Theorem 2.19</u> (H(div)-trace theorem)

There exists a unique continuous linear operator

$$\gamma_n \in L(H(\operatorname{div}), H^{-1/2}(\Gamma))$$

such that

$$\gamma_n q(x) = q(x)\cdot n(x) \quad \dot\forall x \in \Gamma \ \forall q \in H(\operatorname{div}) \cap [C^1(\bar\Omega)]^d,$$

and, $\forall q \in H(\operatorname{div})$ and $\forall v \in H^1(\Omega)$, we have

$v=1$ $$(17)\quad \langle \gamma_n q, \gamma_0 v\rangle_{H^{-1/2}\times H^{1/2}} = \int_\Omega (\operatorname{div} q \cdot v + q \cdot \nabla v) dx$$

<u>Proof:</u> Following the closure principle, it remains to prove <u>continuity</u> (=✝) on a smooth and dense subset: Let $q \in H(\operatorname{div}) \cap [C^1(\bar\Omega)]^d$:

$$(18)\quad \|\gamma_n q\|_{H^{-1/2}(\Gamma)} = \sup_{w \in H^{1/2}(\Gamma)} \frac{\int_\Gamma q\cdot n \cdot w \, ds}{\|w\|_{H^{1/2}(\Gamma)}} \le$$

$$\overset{(17)}{\le} c_e \sup_{v \in H^1(\Omega)} \frac{\int_\Omega (\operatorname{div} q \cdot v + q\cdot\nabla v)dx}{\|v\|_{H^1(\Omega)}} \overset{\text{Cauchy}}{\le} c_e \sup_{v\in H^1(\Omega)} \frac{\|q\|_{H(\operatorname{div})}\|v\|_{H^1(\Omega)}}{\|v\|_{H^1(\Omega)}}$$

↑ (9)e $\|v\|_{H^1(\Omega)} \le c_e \|w\|_{H^{1/2}(\Gamma)}$, $\gamma_0 v = w$

$$= c_e \|q\|_{H(\operatorname{div})}$$

Closure Principle q.e.d.