Proof via "Closure Principle"!

$$W_1^1(\Omega) = \overline{C^1(\bar{\Omega})}^{\|\cdot\|_{W_1^1(\Omega)}} \ni w \xleftarrow[m\to\infty]{W_1^1(\Omega)} w_m \in C^1(\bar{\Omega}).$$

Trace Theorem (→ see $(9)_0$ on T2-07) gives

$$\|\underset{\gamma_0 w}{\underset{\|}{w}} - \underset{\gamma_0 w_m}{\underset{\|}{w_m}}\|_{L_1(\Gamma)} \le c\,\|w - w_m\|_{W_1^1(\Omega)} \xrightarrow[m\to\infty]{} 0$$

The classical formula yields

$$\int_\Omega \partial_i w_m \, dx = \int_\Gamma w_m \cdot n_i \, ds$$

$$\downarrow \qquad\qquad \downarrow \qquad m\to\infty$$

$$\int_\Omega \partial_i w \, dx = \int_\Gamma w \cdot n_i \, ds$$

$$\left| \int_\Gamma w \cdot n_i \, ds - \int_\Gamma w_m n_i \, ds \right| \le \int_\Gamma |w - w_m| \cdot |n_i| \, ds$$

$$\le \underbrace{\|n_i\|_{L_\infty(\Gamma)}}_{\le 1} \|w - w_m\|_{L_1(\Gamma)} \le c\,\|w - w_m\|_{W_1^1(\Omega)} \xrightarrow{n\to\infty} 0$$

q.e.d.

Gauss' Integration Theorem (balance identity):

Let $w = (w_1, \dots, w_d)^T$ a vector field with $w_i \in W_1^1(\Omega)$.
(14) immediately yields Gauss' Integration Theorem:

$$(16) \quad \int_\Omega \operatorname{div} w \, dx = \sum_{i=1}^{d} \int_\Omega \partial_i w_i \, dx = \sum_{i=1}^{d} \int_\Gamma w_i n_i \, ds = \int_\Gamma w \cdot n \, ds$$