Poincaré-type Inequalities:

- **Lemma 2.16:**

Ass.: $1 \le p < \infty$,
$\omega \subset \Omega : |\omega| := \operatorname{meas}\omega := \int_\omega dx > 0$, $* \wedge$ Lip

St.: Then there exists a constant $\bar{c} = const > 0$:

$$(12) \quad \int_\Omega |u|^p dx \le \bar{c}^p \Big\{ \Big|\int_\omega u(x)dx\Big|^p + \int_\Omega |\nabla u|^p dx \Big\} \quad \forall u \in W_p^1(\Omega).$$

Proof:
From Sobolev's norm equivalence Theorem 2,13, we get

$$\|u\|^*_{W_p^1(\Omega)} := \Big(\Big|\int_\omega u dx\Big|^p + |u|^p_{W_p^1(\Omega)} \Big)^{1/p} \simeq \|u\|_{W_p^1(\Omega)}.$$

Indeed, $f_1(u) := \big|\int_\Omega u(x)\,dx\big|$ fulfils the assumption of Theorem 2.13:

1. $f_1(\cdot) : W_p^1(\Omega) \to \mathbb{R}_0^1$ is a semi-norm: (mms)
2. $0 \le f_1(u) = \big|\int_\omega 1 \cdot u\,dx\big| \underset{\text{Hölder}}{\le} |\omega|^{1/q} \big(\int_\omega |u|^p dx\big)^{1/p}$

$\le |\omega|^{1/q} \|u\|_{W_p^1(\Omega)} \quad \forall u \in W_p^1(\Omega)$.

3. $\left.\begin{array}{l} v = c \in P_0 \quad \wedge \\ f_1(c) = \big|\int_\omega c\,dx\big| = |c| \cdot \operatorname{meas}\omega = 0 \end{array}\right\} \Rightarrow v = c = 0.$

q.e.d.

- In the case $\omega = \Omega$, inequality (12) is called Poincaré or Poincare-Friedrichs inequality (cf. Exercise 2.14.1 a) !): $\bar{c} = c_p$