Friedrichs-type Inequalities:

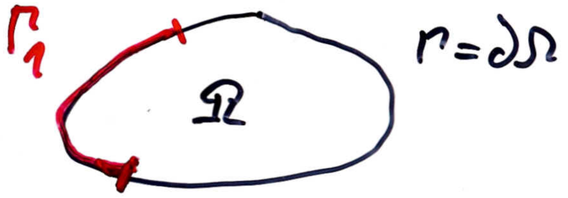
- Let us consider the following setting:

$\Gamma_1 \subset \Gamma = \partial\Omega$: $\text{meas}_{d-1}(\Gamma_1) := \text{meas}_{\mathbb{R}^{d-1}}(\Gamma_1) := \int_{\Gamma_1} ds > 0$,

$V_0 := \{ v \in W_p^1(\Omega) : \gamma_{\Gamma_1} v := v|_{\Gamma_1} = 0 \} \subset W_p^1(\Omega)$,

$V_0 = \overset{\circ}{W}{}_p^1(\Omega)$ if $\Gamma_1 = \Gamma$,

$1 \le p < \infty$



- **Lemma 2.15:**

Ass.: $1 \le p < \infty$; $\Gamma_1 \subset \Gamma$: $\text{meas}_{d-1}(\Gamma_1) = |\Gamma_1| > 0$.

St.: Then there exists a positive constant $\bar{c} = \text{const} > 0$ such that

$$(11) \quad \int_\Omega |u|^p dx \le \bar{c}^p \int_\Omega |\nabla u|^p dx \quad \forall u \in V_0 .$$

In the case $\Gamma_1 = \Gamma$, inequality (11) is also called Friedrichs' inequality: $c_F = \bar{c}$.

Proof: Using Sobolev's norm equivalence Theorem 2.13, we first show that

$$\|u\|^*_{W_p^1(\Omega)} := \left(f_1^p(u) + |u|^p_{W_p^1(\Omega)} \right)^{1/p} \simeq \|u\|_{W_p^1(\Omega)} \text{ in } W_p^1(\Omega)$$

with $f_1(u) := \left(\int_{\Gamma_1} |u|^p ds \right)^{1/p}$.

Indeed, $f_1(\cdot)$ fulfils the assumptions of Th. 2.13:

1) $f_1(\cdot) : W_p^1(\Omega) \longrightarrow \mathbb{R}_0^1 = [0,\infty)$ is a semi-norm: (mms)

2) $0 \le f_1(u) = \left(\int_{\Gamma_1} |u|^p ds \right)^{1/p} \le \left(\int_\Gamma |u|^p ds \right)^{1/p}$

$$= \|u\|_{L_p(\Gamma)} \underset{(9)_0}{\le} c \, \|u\|_{W_p^1(\Omega)} .$$