Exercise 2.14: $\|\cdot\|^* \simeq \|\cdot\|$

Show that the following new norms $\|\cdot\|^*$ are equivalent to the standard norms $\|\cdot\|$ in the corresponding Sobolev spaces:

1. in $W_p^1(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^1(\Omega)}$:

   a) $\|u\|^*_{W_p^1(\Omega)} := \left(\left| \int_\Omega u(x)\,dx \right|^p + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

   (cf. the Poincaré inequality)

   b) $\|u\|^*_{W_p^1(\Omega)} := \left(\left| \int_{\partial\Omega} u(x)\,ds \right|^p + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

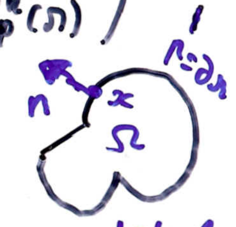
   c) $\|u\|^*_{W_p^1(\Omega)} := \left(\int_{\partial\Omega} |u(x)|^p\,ds + |u|^p_{W_p^1(\Omega)} \right)^{1/p}$

   ...

2. in $W_p^k(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^k(\Omega)}$:

   $\|u\|^*_{W_p^k(\Omega)} := \left(\sum_{\ell=0}^{k-1} \int_{\partial\Omega} \left| \frac{\partial^\ell u}{\partial n^\ell} \right|^p ds + |u|^p_{W_p^k(\Omega)} \right)^{1/p}$,

   where $n = n(x) = (n_1, \dots, n_d)^T$ denotes the exterior unit normal at $x \in \Gamma = \partial\Omega$



3. in $\overset{\circ}{W}{}_p^k(\Omega)$ with $\|\cdot\| := \|\cdot\|_{W_p^k(\Omega)}$:

   Remark: Ass. $\partial\Omega \in C^{0,1}$ (Lip) is not needed!

   $\|u\|^*_{W_p^k(\Omega)} := |u|_{W_p^k(\Omega)}$,

   i.e. in the subspace $\overset{\circ}{W}{}_p^k(\Omega)$ of $W_p^k(\Omega)$, the standard semi-norm $|\cdot|_{W_p^k(\Omega)}$ is a norm that is equivalent to the norm $\|\cdot\|_{W_p^k(\Omega)}$!

Solution: see Tutorial 04

∧ 1a) on the BLACKBOARD!