(11)

2.3. Sobolev's Norm Equivalence Theorem and Some Useful Inequalities (cf. also Th. 1.3)

Theorem 2.13:

Ass.: • $\Omega \subset \mathbb{R}^d$ * ∧ Lip

• $1 \le p < \infty$

• $k = 0, 1, 2, \dots$; $\sigma \in (0,1]$, $s = k + \sigma$

• Let $f_i : W_p^{k+\sigma}(\Omega) \longrightarrow \mathbb{R}^1_0 := [0,\infty)$, $i = 1,\dots,l$

be a system of semi-norms: $\forall i = \overline{1,l}$

a) $\exists c_i = const > 0 : 0 \le f_i(u) \le c_i \|u\|_{W_p^s(\Omega)} \; \forall u \in W_p^s(\Omega)$

b) $\left.\begin{array}{l} f_i(v) = 0 \;\forall i = \overline{1,l} \\ v \in P_k := \{\sum\limits_{|\alpha| \le k} c_\alpha x^\alpha\} \end{array}\right\} \Rightarrow v \equiv 0.$

St.: Then $\exists \underline{c}, \overline{c} = const > 0$:

(10) $$\underline{c}\, \|u\|^*_{W_p^s(\Omega)} \le \|u\|_{W_p^s(\Omega)} \le \overline{c}\, \|u\|^*_{W_p^s(\Omega)} \quad \forall u \in W_p^s(\Omega)$$

$|||\cdot||| = \|\cdot\|^*$ $$\|u\|^*_{W_p^s(\Omega)} := \left(\sum_{i=1}^{l} f_i^p(u) + |u|^p_{W_p^s(\Omega)}\right)^{1/p} \simeq \|u\|^*_{W_p^s(\Omega)},$$

$$\|u\|_{W_p^s(\Omega)} := \left(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u|^p dx + |u|^p_{W_p^s(\Omega)}\right)^{1/p},$$

$$|u|^p_{W_p^{k+\sigma}(\Omega)} := \sum_{|\alpha| = k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{d+p\sigma}} dx\,dy, \quad \sigma \in (0,1),$$

$$|u|^p_{W_p^{k+1}(\Omega)} := \sum_{|\alpha| = k+1} \int_\Omega |\partial^\alpha u(x)|^p dx, \quad \sigma = 1,$$

$|\cdot|_{W_p^{k+\sigma}(\Omega)}$ – standard semi-norm in $W_p^{k+\sigma}(\Omega)$:

$$\operatorname{Ker} |\cdot|_{W_p^{k+\sigma}} = P_k$$

black board **Proof** for $\sigma = 1$ and $p = 2$ on the black board, but the proof is also valid for the general case!

$\|\cdot\|_{W_2^{k+1}} = \|\cdot\|_{k+1}$, $|\cdot|_{W_2^{k+1}(\Omega)} = |\cdot|_{k+1}$ etc.