⑧

The Sobolev-Spaces $\overset{\circ}{W}{}^k_p(\Omega)$, $1 \le p \le \infty$, $k = 0, 1, \ldots$:

- **Def. 2.8:**

$$\overset{\circ}{W}{}^k_p(\Omega) := \overline{\overset{\circ}{C}{}^\infty(\Omega)}^{\|\cdot\|_{W^k_p}} = \text{closure of } \overset{\circ}{C}{}^\infty(\Omega) \text{ in } W^k_p(\Omega).$$

Here we do not need the assumption that Ω is $\in$ Lip!

- **Properties of the spaces $\overset{\circ}{W}{}^k_p(\Omega)$:**

1. $\overset{\circ}{W}{}^k_p(\Omega)$ is a closed subspace of $W^k_p(\Omega)$ since $\overset{\circ}{C}{}^\infty(\Omega)$ is a linear subspace of $W^k_p(\Omega)$.
2. $\overset{\circ}{W}{}^0_p(\Omega) = L_p(\Omega)$ since $\overset{\circ}{C}{}^\infty(\Omega)$ is dense in $L_p(\Omega)$!
3. $\gamma_0 u =: u|_{\Gamma = \partial\Omega} = 0 \quad \forall u \in \overset{\circ}{W}{}^1_p(\Omega)$ (mms)
4. $\gamma_0 \partial^\alpha u = 0 \quad \forall u \in \overset{\circ}{W}{}^k_p(\Omega) \ \forall \alpha : |\alpha| \le k-1$

   $\left(\Rightarrow \gamma_0 u = \gamma_0 \frac{\partial u}{\partial n} = \ldots = \gamma_0 \frac{\partial^{k-1} u}{\partial n^{k-1}} = 0\right)$
5. $\overset{\circ}{H}{}^k(\Omega) = H^k_0(\Omega) = \overset{\circ}{W}{}^k_2(\Omega)$ is a H-space.

The spaces $W^{-k}_q(\Omega)$, $k = 0, 1, 2, \ldots$:

- **Def. 2.9:**

Let $1 < p < \infty$, $\frac{1}{q} + \frac{1}{p} = 1$, $k = 0, 1, 2, \ldots$

$$W^{-k}_q(\Omega) := \{u \in D'(\Omega) : \|u\|_{W^{-k}_q(\Omega)} < \infty\},$$

with the so-called "negative" norm (mms)

$$\|u\|_{W^{-k}_q(\Omega)} := \sup_{\substack{\varphi \in D(\Omega) \\ \varphi \ne 0}} \frac{|\langle u, \varphi \rangle_{D' \times D}|}{\|\varphi\|_{W^k_p(\Omega)}}$$

- **Lemma 2.10:** $W^{-k}_q(\Omega) = [\overset{\circ}{W}{}^k_p(\Omega)]^*$

Ass.: Let $1 < p < \infty$, $q^{-1} + p^{-1} = 1$, $k = 0, 1, 2, \ldots$;
St.: Then the spaces $W^{-k}_q(\Omega)$ and $[\overset{\circ}{W}{}^k_p(\Omega)]^*$ can be identified.

Proof: see Numerik I, pp. 67 - 68. ∎