Sobolev-Spaces $W_p^k(\Omega)$, $1 \le p \le \infty$, $k = 0, 1, \ldots$:

- **Def. 2.7:** $W_p^k(\Omega)$

$$W_p^k(\Omega) := \{ u \in L_p(\Omega) : \exists\, \partial^\alpha u \in L_p(\Omega)\ \forall \alpha : |\alpha| \le k \},$$

with the norm

$$\|u\|_{W_p^k(\Omega)} := \Big(\sum_{|\alpha| \le k} \int_\Omega |\partial^\alpha u(x)|^p dx \Big)^{1/p} \quad \text{for } p < \infty,$$

$$\|u\|_{W_\infty^k(\Omega)} := \max_{|\alpha| \le k} \|\partial^\alpha u\|_{L_\infty(\Omega)}.$$

Notation: $\partial^\alpha u = u$ for $|\alpha| = 0$, i.e. $\alpha = (0, \ldots, 0)$, $W_p^0(\Omega) = L_p(\Omega)$.

- **Properties:**
  1. $W_p^k(\Omega)$ are B-spaces:
     - separable ($\exists$ countable dense subset),
     - uniformely convex for $1 < p < \infty$,
     - reflexive ($X = X^{**}$, $X = W_p^k(\Omega)$) for $1 < p < \infty$.
  2. $H^k(\Omega) = W_2^k(\Omega)$ are H-spaces:

$$\|u\|_{H^k(\Omega)} = \|u\|_k = (u,u)_{H^k(\Omega)}^{0.5} = (u,u)_k^{0.5},$$

$$(u,v)_k := \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u\, \partial^\alpha v\, dx.$$

  3. Define $\tilde{W}_p^k(\Omega) = H_p^k(\Omega) := \overline{C^\infty(\bar\Omega)}^{\|\cdot\|_{W_p^k}}$ in $W_p^k(\Omega)$.

Then

$$\tilde{W}_p^k(\Omega) \underset{\neq}{\subseteq} W_p^k(\Omega) \,!$$

in general

If Ω is $\dagger$ and Lip, then

$$\tilde{W}_p^k(\Omega) = W_p^k(\Omega) = \overline{C^\ell(\bar\Omega)}^{\|\cdot\|_{W_p^k}}$$

for $\ell \ge k$, e.g. $\ell = \infty$. Thus, in the following, we assume Ω is $\dagger$ and Lip.

Note that then we can use the "closure principle"!