③ ■ Distributions:

- **Def. 2.3:** Distributions = gen. functions ; $\mathcal{D}'(\Omega)$

Every <u>linear</u> and <u>continuous</u> <u>functional</u> on $\mathcal{D}(\Omega)$ is called <u>distribution</u> or <u>generalized function</u>.

The set of all distributions (= linear space) is denoted by $\mathcal{D}'(\Omega)$:

$$\langle \cdot , \cdot \rangle = \langle \cdot , \cdot \rangle_{D' \times D} : \ D'(\Omega) \times D(\Omega) \longrightarrow \mathbb{R}^1$$

i.e.

$$u \in D'(\Omega) : \ \varphi \in D(\Omega) \longrightarrow \langle u, \varphi \rangle \in \mathbb{R}^1 :$$

a) linear

b) continuous wrt the "convergence" introduced in $D(\Omega)$:

$$\varphi_n \xrightarrow[n \to \infty]{} 0 \text{ in } D(\Omega) \Rightarrow \langle u, \varphi_n \rangle \xrightarrow[n \to \infty]{} 0$$

- **Examples 2.4:**

1. Let $u \in L_{loc}(\Omega)$, e.g. $u \in L_p(\Omega) < L_{loc}(\Omega)$, $p \geq 1$;
Then the relation

(6) $$\langle \tilde{u}, \varphi \rangle := \int_\Omega u(x) \varphi(x) \, dx \quad \forall \varphi \in D(\Omega)$$

uniquely defines a distribution $\tilde{u} \in D'(\Omega)$ (mms)
that will be identified with $u \in L_{loc}(\Omega)$.
A distribution which has the representation (6) is called <u>regular</u>, otherwise <u>singular</u>.

2. Let $\xi \in \Omega$ be fixed. Then the relation

(7) $$\langle \delta_\xi, \varphi \rangle := \varphi(\xi) \quad \forall \varphi \in D(\Omega)$$

obviously (mms) defines a distribution
$\delta_\xi = \delta(\cdot - \xi) \in D'(\Omega)$ called Dirac's δ-distribution. δ_ξ is a singular distr. (mms*)