Sobolev's Derivatives:

Def. 2.1:
A function $w \in L_{loc}(\Omega) := \{$space of all locally integrable functions in $\Omega\}$ is called αth Sobolev's generalited derivative of some function $u \in L_{loc}(\Omega)$ iff

$$(3) \quad \int_\Omega u\, \partial^\alpha v \, dx = (-1)^{|\alpha|} \int_\Omega w \cdot v \, dx \quad \forall v \in \overset{\circ}{C}{}^\infty(\Omega).$$

Notation: $w = \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$.

- Properties of the generalited (weak) derivatives:
  1. For $u \in C^k(\overline{\Omega})$, the classical and the generalited derivatives coincide up to the order k, i.e. $|\alpha| \le k$!
  2. The generalited derivatives are uniquely defined up to a set of measure zero!
  3. The generalized derivatives preserve almost all properties (e.g. calculus) of the classical derivatives!

Remark 2.2:
The concept of generalited (weak) derivatives can also be extended to other differential operators like

a) div of a vector function $u = (u_1, \ldots, u_d)^T \in [L_{loc}(\Omega)]^d$:

$$(4) \quad w = \operatorname{div} u \in L_{loc}(\Omega): \int_\Omega u^T \nabla \varphi \, dx = -\int_\Omega w \varphi \, dx \quad \forall \varphi \in \overset{\circ}{C}{}^\infty(\Omega).$$

b) curl of a vector function $u = (u_1, \ldots, u_d)^T \in [L_{loc}(\Omega)]^d$:

$$(5) \quad w = \operatorname{curl} u \in [L_{loc}(\Omega)]^d: \int_\Omega u \cdot \operatorname{curl} \varphi \, dx = \int_\Omega w \cdot \varphi \, dx \quad \forall \varphi \in [\overset{\circ}{C}{}^\infty(\Omega)]^d.$$

etc.