2.4.3. Convergence in the H^1-norm

Theorem 2.8: (W_2^1 - Convergence)

Ass.: 1. Standard assumption for variational problems:

1) $F \in V_0^*$ $\quad (V_0 \subset V = W_2^1(\Omega) = H^1(\Omega))$

2) $a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}^1$ ~ continuous bilin. f.:

2a) $a(v,v) \geq \mu_1 \|v\|_1^2 \;\; \forall v \in V_0$,

2b) $|a(u,v)| \leq \mu_2 \|u\|_1 \|v\|_1 \;\; \forall u,v \in V_0$,

2. Ass. 1 and 2 of Th. 2.6 (approx. theorem),

3. $V_{gh} = g_h + V_{0h} \subset V_g$ - finite dim. hyperplane, with $V_{0h} \subset V_0$ - FE subspace, $g_h \in V_g \cap V_h$ given,

4. $u \in V_g : a(u,v) = \langle F, v \rangle \;\; \forall v \in V_0$ (1)

$u_h \in V_{gh} : a(u_h, v_h) = \langle F, v_h \rangle \;\; \forall v_h \in V_{0h}$ $(1)_h$

5. Regularity result:

a) $u \in V_g \cap W_2^{k+1}(\Omega)$

b) $u \in V_g$ and $u \in W_2^{k+1}(\delta_r) \;\; \forall r \in R_h \;\; \forall h \in \circledast$

St.: Then we have the following error estimate:

$$(23) \quad \|u-u_h\|_{1,\Omega} \underset{5b)}{\leq} \underbrace{\frac{\mu_2}{\mu_1} \bar{a}_{1,k+1}}_{=: c_{1,k+1}} \Big[\sum_{r \in R_h} h_r^{2k} |u|^2_{k+1,\delta_r} \Big]^{1/2} \underset{5a)}{\leq} c_{1,k+1} h^k |u|_{k+1,\Omega}$$

Proof follows immediately from (15) = CEA and the approximation Theorem 2.6 resp. Remark 2.7.2 (i.e. (18")):

$$\|u-u_h\|_{1,\Omega} \underset{\text{CEA}}{\leq} \frac{\mu_2}{\mu_1} \inf_{v_h \in V_{gh}} \|u - v_h\|_{1,\Omega} \underset{\text{Th. 2.6}}{\leq} c_{1,k+1} [\ldots]^{1/2} \leq c_{1,k+1} h^k |u|_{k+1,\Omega}$$

q.e.d