3.4.2. The Approximation Theorem

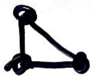
Theorem 3.6:

Ass.: 1. $\bar{\Omega} = \bigcup_r \bar{\delta}_r$: reg. triang. in the sense of Def. 2.3 : $x_{\delta_r}(\cdot) \in P_1$ (8)-(10)

2. $P_k \subset \mathcal{F}(\Delta)$

3a) $u \in H^{k+1}(\Omega)$ or 3b) $u \in H^{k+1}(\delta_r)\ \forall r \in R_h\ \forall h \in \Theta$

(18) **St.:** $\inf\limits_{v_h \in V_h} |u - v_h|_{s,\Omega} \overset{b)}{\le} \tilde{a}_{s,k+1} \Big[\sum\limits_{r \in R_h} h_r^{2(k+1-s)} |u|^2_{k+1,\delta_r} \Big]^{1/2} \overset{a)}{\le} a_{s,k+1} h^{k+1-s} |u|_{k+1,\Omega}$

Proof: $d = 2$, $k = 1$,  $\mathcal{F}(\Delta) = P_1$, $s = 1$ ($s = 0$: ums)

1. $\inf\limits_{v_h \in V_h} |u - v_h|_{1,\Omega} \le |u - \mathrm{int}_{V_h}(u)|_{1,\Omega} = |e_h|_{1,\Omega}$

2. $\delta_r \to \Delta$

$$|e_h|^2_{1,\Omega} \le \bar{c}_1 \tilde{c}_3^2 \sum_{r \in R_h} h_r^{d-2} \int_\Delta |\nabla_\xi e_h(x_{\delta_r}(\xi))|^2 d\xi \le$$

3. B&H-Lemma 2.17: $|e_h(x_\delta(\xi))|_{1,\Delta} \le c_B\, |u(x_{\delta_r}(\xi))|_{2,\Delta}$ (20)

$$\le \bar{c}_1 \tilde{c}_3^2 c_B^2 \sum_{r \in R_h} h_r^{d-2} \int_\Delta \sum_{|\alpha|=2} |\partial_\xi^\alpha u(x_{\delta_r}(\xi))|^2 d\xi$$

4. Return mapping $\Delta \to \delta_r$:

$$\le \underbrace{\bar{c}_1 \underline{c}_1^{-1} \tilde{c}_3^2 c_B^2 c_4^4}_{=: \tilde{a}_{1,2}^2} \sum_{r \in R_h} h_r^{d-2-d+4} \int_{\delta_r} \sum_{|\alpha|=2} |\partial_x^\alpha u|^2 dx$$

(q.e.d)

It remains to prove estimate (20) by B&H-Lemma ~~2.5~~ 2.17 :

$$\ell(u) := \int_\Delta \nabla_\xi^T w \cdot \nabla_\xi \Big(\overbrace{\underbrace{u(x_\delta(\xi))}_{=: u(\xi)} - \underbrace{\mathrm{int}_{V_h}\big(u(x_{\delta_r}(\xi))\big)}_{=: \mathrm{int}_{\mathcal{F}(\Delta)}(u(\xi))}}^{= e_h(x_{\delta_r}(\xi))} \Big) d\xi$$

$\| \ \ell(u)$

$\Longrightarrow$ 1) $\ell(\cdot)$ is linear ✓

2) $\ell(\cdot)$ is bounded on $H^2(\Delta)$ $\Big\}$ $\Rightarrow$ $\ell \in [H^2(\Omega)]^*$

3) $\ell(q) = 0\ \forall q \in P_1$ ✓

$\to$ ✗ (20)