2.4.2. The Approximation Theorem

- **Theorem 2.6:** (approximation theorem)

Ass.: 1. Let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be provided with a regular triangulation in the sense of Def. 2.3, i.e.
$\forall h \in \Theta: \overline{\Omega} = \bigcup_{r \in R_h} \overline{\delta_r}$, $\delta_r \underset{\xi = \xi_{\delta_r}(x)}{\overset{x = x_{\delta_r}(\xi)}{\rightleftarrows}} \Delta \ \forall r \in R_h$:

$h \rightsquigarrow h_r$

(8) $\tilde{\underline{c}}_1 h_r^d \leq |J_{\delta_r}| \leq \overline{c}_1 h_r^d \quad \forall \xi \in \Delta \ \forall r \in R_h$,

(9) $\| J_{\delta_r} \| := \sqrt{\lambda_{max}(J_{\delta_r}^T J_{\delta_r})} \leq c_2 h_r$ — " — ,

(10) $\| J_{\delta_r}^{-T} \| \leq \tilde{c}_3 h_r^{-1}$ — " — ,

where, for the time being, $x_{\delta_r}(\cdot) \in P_1(\Delta)$, i.e. an affine linear mapping (see Remark 2.7 for generalization),

$A \rightsquigarrow A_r$, $\mathcal{F} \rightsquigarrow \mathcal{F}_r$

2. $\mathcal{F}(\Delta) = \operatorname{span}\{p^{(\alpha)}(\xi) : \alpha \in A\} \supset P_k(\Delta)$

3. $u \in W_2^{k+1}(\Omega)$, or more general, $u \in W_2^{k+1}(\delta_r) \ \forall r \in R_h \ \forall h \in \Theta$
a) $\Longrightarrow$ b)

St.: $\exists\ \tilde{a}_{s,k+1} = \text{const} > 0$ (independent of h and u)

$$(18) \quad \inf_{v_h \in V_h} \underbrace{|u - v_h|_{s,\Omega}}_{W_2^s(\Omega)} \overset{b)}{\leq} \tilde{a}_{s,k+1} \Big[\sum_{r \in R_h} h_r^{2(k+1-s)} |u|^2_{k+1,\delta_r} \Big]^{1/2}$$

$$\overset{b)}{\leq} a_{s,k+1} h^{k+1-s} \Big[\sum_{r \in R_h} |u|^2_{k+1,\delta_r} \Big]^{1/2}$$

$$\overset{a)}{=} a_{s,k+1} h^{k+1-s} |u|_{k+1,\Omega},$$

where $s = 0, 1$, or $s \in [0, 1]$ (space interpolation theory)