2.4. A-priori Discretization Error Estimates

- **Starting point:**

= Cea's theorem (lemma) ≈ Th. 1.6 NuPDE
with $V = H^1(\Omega) = W_2^1(\Omega)$, $\|\cdot\| = \|\cdot\|_1 = \|\cdot\|_{H^1(\Omega)}$, i.e.
for scalar 2nd-order PDEs:

$$\Rightarrow \underbrace{\| \overset{(1)}{u} - \overset{(1)_h}{u_h} \|_1}_{\text{discretization error}} \le \frac{\mu_2}{\mu_1} \underbrace{\inf_{v_h \in \tilde{V}_{gh}} \| u - v_h \|_1}_{\text{approximation error}} \quad (15)$$

Proof is based on the so-called GALERKIN "orthogonality":

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in \tilde{V}_{0h}$$

- **Road Map for the Proof:**

• Choose

$$v_h = \underset{\substack{\uparrow \\ \text{interpolant}}}{\mathrm{int}_{\tilde{V}_h}}(u) = \sum_{i \in \bar{\omega}_h} u(x^{(i)}) \, p^{(i)}(x) =$$

$u \in V_g \cap C(\bar{\Omega})$, e.g. due to Sobolev's embedding theorem

$$\boxed{\begin{array}{l} W_p^{k+1}(\Omega) \hookrightarrow C(\bar{\Omega}) \\ \text{if } (k+1)\,p > d \end{array}}$$

we have for $d = 2$
$(\Omega \subset \mathbb{R}^d : \text{*}, \text{Lip})$:
$u \in \tilde{V}_g \cap H^2 \Rightarrow u \in V_g \cap C(\bar{\Omega})$
$\uparrow$
$2 \times 2 > d = 2$

$$= \underbrace{\sum_{i \in \omega_h} u(x^{(i)}) p^{(i)}(x)}_{\substack{= 0 \text{ on } \Gamma_1 \\ \approx}} + \underbrace{\sum_{i \in \gamma_h} \underset{g_1(x^{(i)})}{\underset{\shortparallel}{u(x^{(i)})}} p^{(i)}(x)}_{\substack{= g_1(x)\ \forall x \in \Gamma_1 \\ \approx}}$$

? crime ? see Sect. 2.5 !

or, if $u \notin C(\bar{\Omega})$, then choose

$v_h = \mathrm{int}_{\tilde{V}_h}(\text{averaging}(u)) = \mathrm{int}_{\tilde{V}_h}\left(\frac{1}{|\pi_h|} \int_{\pi_h} \tilde{u}(x+\xi)\,d\xi\right)$, $\tilde{u} = Eu$

$$\Rightarrow \underbrace{\inf_{v_h \in \tilde{V}_{gh}} \| u - v_h \|_1}_{\text{approximation error}} \overset{\uparrow}{\le} \underbrace{\| u - \mathrm{int}_{\tilde{V}_h}(u) \|_1}_{\text{interpolation error}} = \| \bullet \|_1 \le \ ?\ \le\ ?\, h^{?}\ ?$$

$(\downarrow)$ $(h \to 0)$

$v_h = \mathrm{int}_{\tilde{V}_h}(u) \in \tilde{V}_{gh}$ $\quad (? \approx ?)$