2.2. The Generation of the FEM-GALERKIN-Scheme: Linear Triangular Elements as Typical Example

2.2.1. A Modell Problem

- Let us consider a 2D heat conduction problem in variational formulation as a model problem:

(2)
$$\boxed{\begin{array}{l}\text{Find } u \in V_g := \{ v \in V = H^1(\Omega) : v = g_1 \text{ on } \Gamma_1 \} : \\ a(u,v) = \langle F, v \rangle \quad \forall v \in V_0 := \{ v \in V : v = 0 \text{ on } \Gamma_1 \}, \\ \text{where } a(u,v) := \int_\Omega [\lambda(x) \nabla^T u \cdot \nabla v + a(x) u v] \, dx + \int_{\Gamma_3} \varkappa u v \, ds, \\ \qquad \langle F, v \rangle := \int_\Omega f v \, dx + \int_{\Gamma_2} g_2 v \, ds + \int_{\Gamma_3} g_3 v \, ds, \\ \qquad \Omega \subset \mathbb{R}^2 \ast, \ \Gamma = \partial\Omega \in C^{1,1}, \ \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 = \Gamma. \end{array}}$$

under the following assumptions: a.e. in Ω

(3)
$$\begin{cases} 1) \ \lambda \in L_\infty(\Omega) : \exists \underline{\lambda}, \overline{\lambda} = \text{const} > 0 : \underline{\lambda} \le \lambda(x) \le \overline{\lambda} \ \dot{\forall} x \in \Omega \\ 2) \ a \in L_\infty(\Omega) : a(x) \ge 0 \ \dot{\forall} x \in \Omega, \\ 3) \ \varkappa \in L_\infty(\Gamma_3) : \varkappa(x) \ge 0 \ \dot{\forall} x \in \Gamma_3, \\ 4) \ f \in L_2(\Omega) \\ 5) \ g_2 \in L_2(\Gamma_2), \ g_3 \in L_2(\Gamma_3), \\ 6) \ g_1 \in H^{1/2}(\Gamma_1), \text{ i.e. } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1, \\ 7) \ \Omega \subset \mathbb{R}^2 \ast, \ \Gamma = \partial\Omega \in C^{0,1}, \ \text{meas}_1(\Gamma_1) > 0. \end{cases}$$

- $\boxed{\text{Ex. 2.1.}}$ Give the classical formulation (CF) of (2) under classical smoothness assumptions, and give the CF of the example CHIP (interface conditions!)!

- $\boxed{\text{Ex. 2.2.}}$ Show that the assumptions of the LAX-MILGRAM-Theorem are fulfilled under the assumptions (3) imposed on the data!
<u>Hint:</u> First homogenize the Dirichlet BC!
<u>Remarks</u> $\Rightarrow \exists! \, u \in V_g : (2)$ ∎