- Let us now consider the DIRICHLET problem for the POISSON equation as an EXAMPLE:

$(30)_{CF}$ $\quad -\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\Gamma = \partial\Omega$

$\Rightarrow V = H^1(\Omega)$, $V_g = V_0 = \overset{\circ}{H}{}^1(\Omega)$ (since $g = 0$)

$(30)_{MP}$ Find $u \in V_0$: $E(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v\, dx \to \min!$

$\Updownarrow$

$(30)_{VP}$ Find $u \in V_0$: $\int_\Omega \nabla^T u \nabla v\, dx = \int_\Omega f v\, dx \quad \forall v \in V_0$

$\Updownarrow$

$(30)_{DVP}$ Find $w \in W$: $G(w) := -\frac{1}{2}\int_\Omega |\nabla w|^2 dx \to \max!$

$W := \{w \in H^1(\Omega) : \int_\Omega \nabla^T w \nabla v\, dx = \int_\Omega f v\, dx \ \forall v \in V_0\}$

$= \{w \in H^1(\Omega) : -\operatorname{div} \nabla w = f$ in the weak sense$\}$

! no BC !

$\Updownarrow \quad \boxed{\sigma = \nabla u}$

$(30)_{DP}$ Find $\sigma \in W(f)$: $G(\sigma) = \sup_{\tau \in W(f)} G(\tau)$

with $G(\tau) := -\frac{1}{2}\int_\Omega |\tau|^2 dx$,

$W(f) := \{\tau \in H(\operatorname{div}, \Omega) : -\operatorname{div} \tau = f \text{ in } \Omega\}$

Remarks:

1. $(30)_{MP/VP}$: $V_0 \ (= V_g) \leadsto$ essential BC must be fulfilled!

   $(30)_{DVP/DP}$: W or $W(f) \leadsto$ PDE must be fulfilled in a weak sense!

2. primal $\longrightarrow$ mixed $\longleftarrow$ dual

   Lagrange-functional

   $L(\tau, q) = \frac{1}{2}\int_\Omega |\tau|^2 dx + \int_\Omega q\,(\operatorname{div}\tau + f)\, dx$