Solid Mechanics:

3) Dirichlet problem for Poisson's eqn ($\hat{=}$ elasticity)

- <u>Idea:</u> Hellinger – Reisner

(25)
$$\boxed{\begin{array}{l} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \Gamma \\ \Omega \subset \mathbb{R}^d \ * \end{array}} \longrightarrow \boxed{\begin{array}{l} \sigma - \nabla u = 0 \text{ in } \Omega \\ \operatorname{div} \sigma = -f \text{ in } \Omega \\ u=0 \text{ on } \Gamma = \partial\Omega \end{array}} \begin{array}{l} \int_\Omega \cdot\, \tau \, dx \ \forall \tau \in V \\ \int_\Omega \cdot\, q \, dx \ \forall q \in Q \end{array}$$

← nat. BC !!!

- $\int_\Omega \sigma^T \tau \, dx + \int_\Omega u \cdot \operatorname{div} \tau - \int_\Gamma u \cdot \tau^T n \, ds = 0 \ \forall \tau \in V,$
- $\int_\Omega \operatorname{div} \sigma \cdot q \, dx = -\int_\Omega f \cdot q \, dx \ \forall q \in Q.$

- <u>Spaces:</u> $V = H(\operatorname{div}, \Omega) := \{\tau \in [L_2(\Omega)]^d : \operatorname{div} \tau \in L_2(\Omega)\},$

  with $\|\tau\|_V^2 := \sum_{i=1}^d \|\tau_i\|_{L_2(\Omega)}^2 + \|\operatorname{div}\tau\|_{L_2(\Omega)}^2,$

  $w = \operatorname{div}\tau \in L_2(\Omega)$:

  $\int_\Omega \tau^T \nabla\varphi \, dx = -\int_\Omega w \cdot \varphi \, dx \ \forall \varphi \in \overset{\circ}{H}{}^1(\Omega)$

  $Q = L_2(\Omega)$ with $\|q\|_Q := \|q\|_{L_2(\Omega)}$.

- <u>Mixed Variational Formulation:</u> $f \in L_2(\Omega)$ given

$(25)_{MVF}$

$$\boxed{\begin{array}{llll} \text{Find } \sigma \in V \text{ and } u \in Q: & & & \\ \int_\Omega \sigma^T \tau \, dx & + \int_\Omega u \cdot \operatorname{div}\tau \, dx & = 0 & \forall \tau \in V \\ a(\sigma,\tau) & + b(\tau,u) & = \langle F,\tau\rangle & \\ \int_\Omega q \cdot \operatorname{div}\sigma \, dx & & = -\int_\Omega f \cdot q \, dx & \forall q \in Q \\ b(\sigma,q) & & = \langle G,q\rangle & \end{array}}$$

- <u>Advantages!</u> ⊕ $u=0$ on Γ is now a natural BC!
  - ⊕ $\sigma = \nabla u$ (stresses) are often more interesting than u
  - ⊕ later: Galerkin $V_h \subset V$, $Q_h \subset Q$ ↝ mixed FEM!
- <u>Disadvantages!</u> ⊖ scalar PDE of 2nd order
  $\Rightarrow$ system of $(d+1)$ PDEs of 1st order!
  - ⊖ discrete problems are indefinite!