

1.2.2. BVP for 2nd-Order Elliptic PDE Systems: The Linear Elasticity Problem

■ Classical Formulation of the Linear 3d elasticity problem (see Lectures on MathModTech, Sec. 2.2)

(8) Find displacement field $u(x) = (u_1(x), u_2(x), u_3(x))^T \in X$:

1) equilibrium of forces: $-\text{div } \sigma = f$

$$-\sum_{j=1}^3 \frac{\partial}{\partial x_j} \sigma_{ji}(u(x)) = f_i(x) \quad \forall x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, i=1,2,3$$

|| \leftarrow equilibrium of momentum: $\sigma = \sigma^T$

Kinematics

2) Material Law = Hook's Law: $\sigma = \mathbb{D} \varepsilon$

$$\sigma_{ij} = \sum_{k,l=1}^3 D_{ijkl} \varepsilon_{kl} = \lambda \left(\sum_{k=1}^3 \varepsilon_{kk} \right) \delta_{ij} + 2\mu \varepsilon_{ij}$$

21 independent elast. coeff. of \mathbb{D}

isotropic material:

$$D_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

with Lamé's const $\lambda, \mu = \text{const} > 0$, where $\lambda \neq \lambda(x), \mu \neq \mu(x)$ for hom. mat.

3) Geometrical Strain-Displacement rel.: $\varepsilon = \frac{1}{2} (\nabla u + \nabla u^T)$

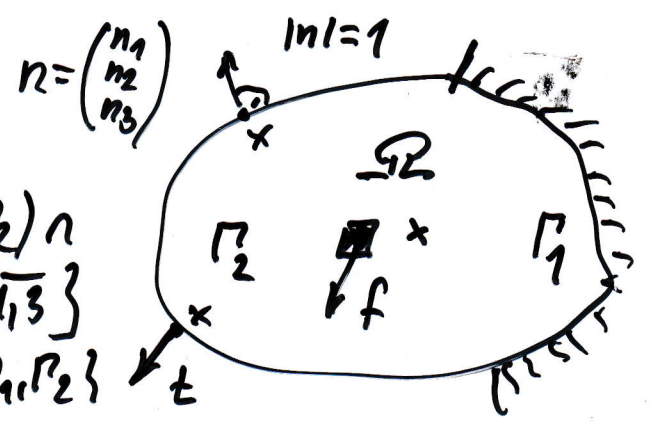
Kinematics $\varepsilon_{ij} = \varepsilon_{ij}(u) := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \varepsilon_{ji} \quad \forall i,j = \overline{1,3}$

4) Boundary Conditions:

1st Kind: $u(x) = 0$ (or $u(x) = \bar{u}(x)$) $\forall x \in \Gamma_1 = \Gamma_D = \Gamma_u$

2nd Kind: $\sum_{j=1}^3 \sigma_{ij}(u(x)) n_j(x) = t_i(x) \quad \forall x \in \Gamma_2 = \Gamma_N = \Gamma_t$

where $X := \{ v = (v_1, v_2, v_3)^T : v_i \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2) \cap C^1(\Omega \cup \Gamma_1), i = \overline{1,3} \}$



input data $\{ D_{ijkl}, f, t, \bar{u}, \Omega, \Gamma_1, \Gamma_2 \}$ sufficiently smooth!