1.2.2. BVP for 2nd-Order Elliptic PDE Systems: The Linear Elasticity Problem

- Classical Formulation of the Linear 3d elasticity problem (see Lectures on MathModTech, Sec. 2.2)

(8) Find displacement field $u(x) = (u_1(x), u_2(x), u_3(x))^T \in X$:

1) equilibrium of forces: $-\operatorname{div} \sigma = f$

Kinetics

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j} \underbrace{\sigma_{ji}}_{=\sigma_{ij}}(u(x)) = f_i(x) \quad \forall x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \ i = 1,2,3$$

equilibrium of momentum: $\sigma = \sigma^T$

2) Material Law = Hook's Law: $\sigma = D\varepsilon$

$$\sigma_{ij} = \sum_{k,l=1}^{3} D_{ijkl}\, \varepsilon_{kl} = \lambda \Big(\sum_{k=1}^{3} \varepsilon_{kk} \Big) \delta_{ij} + 2\mu\, \varepsilon_{ij}$$

21 independent elast. coeff. of 81

isotropic material:
$D_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
with Lamé's const $\lambda, \mu = \text{const} > 0$,
where $\lambda \neq \lambda(x)$, $\mu \neq \mu(x)$ for hom. mat.

3) Geometrical Strain-Displacement rel.: $\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u)$

Kinematics

$$\varepsilon_{ij} = \varepsilon_{ij}(u) := \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \Big) = \varepsilon_{ji} \quad \forall i,j = \overline{1,3}$$

4) Boundary Conditions:

1st Kind: $u(x) = 0$ (or $u(x) = \bar{u}(x)$) $\forall x \in \Gamma_1 = \Gamma_D = \Gamma_u$

2nd Kind: $\sum_{j=1}^{3} \sigma_{ij}(u(x))\, n_j(x) = t_i(x) \quad \forall x \in \Gamma_2 = \Gamma_N = \Gamma_t$

where

$X := \{ v = (v_1, v_2, v_3)^T :$
$v_i \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2) \cap$
$\cap\, C(\Omega \cup \Gamma_1),\ i = \overline{1,3} \}$

input data $\{D_{ijkl}, f, t, \bar{u}, \Omega, \Gamma, \Gamma_1, \Gamma_2\}$ sufficiently smooth!

$n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, $|n| = 1$