Approximation + Stabilität ⇒ diskrete Konvergenz

Btr. Wärmeleitgleichung (A-M):

$u(x,t) := T(x,t)$:

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \underset{\varkappa = \lambda / \rho c}{\varkappa} \dfrac{\partial^2 u}{\partial x^2}(x,t) = \underset{0}{f(x,t)} & \forall x \in (\underset{0}{a}, \underset{L}{b}) \\ & \forall t \in (0, t_E) \\ \text{RB: } u(a,t) = u_a(t),\ u(b,t) = u_b(t), & \forall t \in (0,t_E) \\ \text{AB: } u(x,0) = u_0(x),\ \forall x \in [a,b], & \forall x \in [a,b] \end{cases}$$

Differenzenschema:

$v(x,t) \approx u(x,t) \qquad x = x_i = a + ih,\ i = \overline{0,n},\ h = \Delta x = \frac{b-a}{n}$

$v_i^j = v(x_i,t_j) \qquad t = t_j = j\tau,\ j = \overline{0,m},\ \tau = \Delta t = t_E/m$

$$(2)\quad \begin{cases} \underbrace{\dfrac{v_i^{j+1} - v_i^j}{\tau}}_{=:\, v_{t,i}^j} - \varkappa \underbrace{\dfrac{v_{i-1}^j - 2v_i^j + v_{i+1}^j}{h^2}}_{=:\, v_{\bar{x}x,i}^j} = f_i^j := f(x_i,t_j) & i = \overline{1,n-1},\ j = \overline{0,m-1} \\ \text{RB: } v_0^j = u_a(t_j),\ v_n^j = u_b(t_j), & j = 0,1,\dots,m \\ \text{AB: } v_i^0 = u_0(x_i), & i = \overline{0,n} \end{cases}$$

Fehlerschema:

$z_i^j = z(x_i,t_j) := \overset{(1)}{u(x_i,t_j)} - \overset{(2)}{v(x_i,t_j)}$

$$z_{t,i}^j - \varkappa z_{\bar{x}x,i}^j = u_{t,i}^j - \varkappa u_{\bar{x}x,i}^j - \underbrace{\left(v_{t,i}^j - \varkappa v_{\bar{x}x,i}^j\right)}_{\overset{(2)}{=} f_i^j = f(x_i,t_j) \overset{(1)}{=} \dots} =$$

$$= \frac{u(x_i,t_j+\tau) - u(x_i,t_j)}{\tau} - \varkappa \frac{u(x_i-h,t_j) - 2u(x_i,t_j) + u(x_i+h,t_j)}{h^2}$$

$$- \underbrace{\left[\frac{\partial u}{\partial t}(x_i,t_j) - \varkappa \frac{\partial^2 u}{\partial x^2}(x_i,t_j)\right]}_{= f(x_i,t_j)} = \psi_i^j$$