ÜBUNGEN ZU

NUMERIK PARTIELLER DIFFERENTIALGLEICHUNGEN

für den 18. 1. 2006

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43. Consider the initial-value problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(0) &= u_0 \end{aligned}$$

under the assumption that

$$\|f(t, w) - f(t, v)\| \leq L \|w - v\| \quad \text{for all } t,\ v,\ w.$$

Show that, for each t_j and u_j, there exists a unique solution u_{j+1} of the equation

$$u_{j+1} = u_j + \tau\, f(t_j + \tau, u_{j+1}),$$

if $\tau < 1/L$.

Hint: Apply Banach's fixed point theorem.

44. Consider the initial-value problem

$$\begin{aligned} u'(t) &= f(t, u(t)), \\ u(0) &= u_0 \end{aligned}$$

under the assumption that

$$\|f(t, w) - f(t, v)\| \leq L \|w - v\| \quad \text{for all } t,\ v,\ w,$$

and

$$(f(t, w) - f(t, v), w - v) \leq 0 \quad \text{for all } t,\ v,\ w.$$

Show that, for each $\tau > 0$, t_j and u_j, there exists a unique solution u_{j+1} of the equation

$$u_{j+1} = u_j + \tau\, f(t_j + \tau, u_{j+1}).$$

Hint: Apply Banach's fixed point theorem to the following equivalent equation

$$u_{j+1} = G(u_{j+1}) \equiv (1 - \rho)\, u_{j+1} + \rho\, (u_j + \tau\, f(t_j + \tau, u_{j+1}))$$

for some parameter $\rho \in (0, 1)$. Estimate

$$\|G(w) - G(v)\|^2 = (G(w) - G(v), G(w) - G(v))$$

and choose $\rho \in (0, 1)$ such that G is contractive.

45. Show for the consistency error of the implicit Euler method, given by

$$\psi_\tau(t+\tau) = \frac{1}{\tau}\Big(u(t+\tau) - u(t)\Big) - f(t+\tau, u(t+\tau)),$$

where $u(t)$ denotes a solution of the differential equation

$$u'(t) = f(t, u(t)),$$

the following estimation:

$$\|\psi_\tau(t+\tau)\| \leq \int_t^{t+\tau} \|u''(s)\|\, ds.$$

46. Assume the notations and assumptions of problem 44. Show for the implicit Euler method:

$$u_{j+1} = u_j + \tau\, f(t_{j+1}, u_{j+1})$$

that

$$u(t_{j+1}) = u(t_j) + \tau\, f(t_{j+1}, u(t_{j+1})) + \tau\, \psi_t(t_{j+1}),$$

where $u(t)$ denotes the exact solution of the initial-value problem, and

$$e_{j+1} = e_j + \tau\, \left(f(t_{j+1}, u(t_{j+1})) - f(t_{j+1}, u_{j+1})\right) + \tau\, \psi_\tau(t_{j+1}),$$

where $e_k \equiv u(t_k) - u_k$ denotes the global error.

47. Assume the notations and assumptions of problem 46. Show the following estimation:

$$\|e_{j+1}\| \leq \|e_j\| + \tau\, \|\psi_\tau(t_{j+1})\|.$$

Hint: Multiply the identity

$$e_{j+1} = e_j + \tau\, \left(f(t_{j+1}, u(t_{j+1})) - f(t_{j+1}, u_{j+1})\right) + \tau\, \psi_\tau(t_{j+1})$$

by $e_{j+1} = u(t_{j+1}) - u_{j+1}$ and use Cauchy' inequality for the right hand side.

48. Assume the notations and assumptions of problem 46. Show:

$$\|u(t_j) - u_j\| \leq \tau \int_0^{t_j} \|u''(s)\|\, ds.$$