ÜBUNGEN ZU

NUMERIK PARTIELLER DIFFERENTIALGLEICHUNGEN

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Let $\mathcal{T}_l$ be a subdivision of the interval $\Omega = (0,1)$, given by the nodes

$$0 = x_{l,0} < x_{l,1} < \ldots < x_{l.N_l} = 1.$$

Let $\mathcal{T}_{l+1}$ be the refined subdivision which is constructed from the subdivision $\mathcal{T}_l$ by adding the midpoints of the sub-intervals of $\mathcal{T}_l$. If the nodes are numbered from left to right, we obviously have for the nodes $x_{l+1,i}$ of $\mathcal{T}_{l+1}$:

$$\begin{aligned} x_{l+1,2i} &= x_{l,i}, \\ x_{l+1,2i+1} &= \frac{1}{2}(x_{l,i} + x_{l,i+1}). \end{aligned}$$

From this the following relation between the nodal basis functions of the Courant element with respect to the two subdivisons $\mathcal{T}_l$ and $\mathcal{T}_{l+1}$ can be easily derived for $0 < i < N_l$:

$$\varphi_{l,i}(x) = \frac{1}{2}\,\varphi_{l+1,2i-1}(x) + \varphi_{l+1,2i}(x) + \frac{1}{2}\,\varphi_{l+1,2i+1}(x).$$

36. Let w_l be a finite element function on the coarse grid $\mathcal{T}_l$. Then

$$w_l(x) = \sum_{i=0}^{N_l} w_{l,i}\,\varphi_{l,i}(x).$$

Find a representation of this function with the help of the basis functions of the fine grid $\mathcal{T}_{l+1}$: Compute the coefficients $w_{l+1,i}$ such that

$$w_l(x) = \sum_{i=0}^{N_{l+1}} w_{l+1,i}\,\varphi_{l+1,i}(x).$$

Represent the relation between the coefficient vectors $\underline{w}_{l+1} = (w_{l+1,i})_{i=0,\ldots,N_{l+1}}$ and $\underline{w}_l = (w_{l,i})_{i=0,\ldots,N_l}$ in the form

$$\underline{w}_{l+1} = I_l^{l+1}\underline{w}_l$$

with an appropriate $N_{l+1} \times N_l$ matrix I_l^{l+1}.

37. Let $R : H^1(0,1) \longrightarrow \mathbb{R}$ be a continuous linear functional. The evaluation of R for some finite element function v_{l+1} on the fine grid $\mathcal{T}_{l+1}$ can be represented in the following form:

$$\langle R, v_{l+1}\rangle = \sum_{i=0}^{N_{l+1}} r_{l+1,i}\,v_{l+1,i} = (\underline{r}_{l+1}, \underline{v}_{l+1})_{\ell_2}$$

with

$$\underline{r}_{l+1} = (r_{l+1,i})_{i=0,\ldots,N_{l+1}}, \quad \text{and} \quad r_{l+1,i} = \langle R, \varphi_{l+1,i} \rangle.$$

Find a representation of the evaluation of this functional for a finite element function v_l, defined on the coarse grid $\mathcal{T}_l$ in the form

$$\langle R, v_l \rangle = \sum_{i=0}^{N_l} r_{l,i}\, v_{l,i} = (\underline{r}_l, \underline{v}_l)_{\ell_2}.$$

Show the following relation between the coefficient vectors $\underline{r}_{l+1} = (r_{l+1,i})_{i=0,\ldots,N_{l+1}}$ and $\underline{r}_l = (r_{l,i})_{i=0,\ldots,N_l}$

$$\underline{r}_l = I_{l+1}^l \underline{r}_{l+1} \quad \text{with} \quad I_{l+1}^l = \left(I_l^{i+1}\right)^T.$$

Hint:

$$(\underline{r}_l, \underline{v}_l)_{\ell_2} = \langle R, v_l \rangle = (\underline{r}_{l+1}, \underline{v}_{l+1})_{\ell_2} = (\underline{r}_{l+1}, I_l^{l+1} \underline{v}_l)_{\ell_2}$$

38. Show the following relation

$$K_l = I_{l+1}^l K_{l+1} I_l^{l+1} = \left(I_l^{l+1}\right)^T K_{l+1} I_l^{l+1}.$$

where K_l and K_{l+1} denote the stiffness matrices on the coarse grid $\mathcal{T}_l$ and on the fine grid $\mathcal{T}_{l+1}$, respectively.

Hint: For finite element functions w_l und v_l on the coarse grid we have

$$a(w_h, v_h) = (K_l \underline{w}_l, \underline{v}_l)_{\ell_2}.$$

On the other hand, these finite element functions are also finite element functions on the fine grid, hence

$$a(w_h, v_h) = (K_{l+1} \underline{w}_{l+1}, \underline{v}_{l+1})_{\ell_2}.$$

39. Write a function `RefineUniform(↓coarsemesh,↑finemesh)`, which computes the refined subdivision `finemesh` $= \mathcal{T}_{l+1}$ from a coarse subdivision `coarsemesh` $= \mathcal{T}_l$ as described above.

40. Write a function `Prolongate(↓coarsevector,↑finevector)` for computing $\underline{w}_{l+1} = I_l^{l+1} \underline{w}_l$ with `finevector` $= \underline{w}_{l+1}$ and `coarsevector` $= \underline{w}_l$.

    Write a function `Restrict(↓finevector,↑coarsevector)` for computing $\underline{r}_l = I_{l+1}^l \underline{r}_{l+1}$ with `finevector` $= \underline{r}_{l+1}$ and `coarsevector` $= \underline{r}_l$.

41. Implement the MDS preconditioner for a hierarchy of L grids $\mathcal{T}_1, \ldots, \mathcal{T}_L$:

    (a) If there is only one grid $\mathcal{T}_1$ ($L = 1$), then the MDS preconditioner coindices with the Jacobi preconditioner, i.e.:

$$\underline{w}_1 = D_1^{-1} \underline{r}_1.$$

(b) For a hierarchy of two grids $\mathcal{T}_1$, $\mathcal{T}_2$ ($L = 2$) the correction $\underline{w}_2$, obtained by the MDS preconditioner for a given residual $\underline{r}_2$, is the sum of the correction, obtained by the Jacobi preconditioner on the fine grid for the residual $\underline{r}_2$, and the (prolongated) correction, obtained by the Jacobi preconditioner on the coarse grid for the (restricted) residual $\underline{r}_1$, i.e.:

$$\underline{w}_2 = D_2^{-1}\underline{r}_2 + I_1^2\underline{w}_1$$

with

$$\underline{w}_1 = D_1^{-1}\underline{r}_1 \quad \text{for} \quad \underline{r}_1 = I_2^1\underline{r}_2.$$

(c) For a hierarchy of L grids $\mathcal{T}_1, \ldots, \mathcal{T}_L$ the correction $\underline{w}_L$, obtained by the MDS preconditioner for a given residual $\underline{r}_L$, is the sum of the correction, obtained by the Jacobi preconditioner on the grid $\mathcal{T}_L$ for the residual $\underline{r}_L$, and the (prolongated) correction, obtained by the MDS preconditioner on the grid $\mathcal{T}_{L-1}$ for the (restricted) residual $\underline{r}_{L-1}$, i.e.:

$$\underline{w}_L = C_L^{-1}\underline{r}_L = D_L^{-1}\underline{r}_L + I_{L-1}^L\underline{w}_L$$

with

$$\underline{w}_{L-1} = C_{L-1}^{-1}\underline{r}_{L-1} \quad \text{für} \quad \underline{r}_{L-1} = I_L^{L-1}\underline{r}_L.$$

Hint: Use a recursive function of the following form:

```
MDS(l,r,w) {
  ...
  if (l == 1)
    w = JacobiPreconditioner.solve(l,r);
  else {
    w = JacobiPreconditioner.solve(l,r);
    Restrict(r,r_coarse);
    MDS(l-1,r_coarse,w_coarse);
    Prolongate(w_coarse,w_fine);
    w += w_fine;
  }
};
```