ÜBUNGEN ZU

NUMERIK PARTIELLER DIFFERENTIALGLEICHUNGEN

für den 30. 11. 2005

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26. Let K be a symmetric and positive definite matrix, let $u^{(n)}$ be an approximation of the exact solution u of the linear system

$$Ku = f,$$

and let $p^{(n)}$ be a given search direction. Determine the parameter $\alpha^{(n)} \in \mathbb{R}$ such that the next approximation of the form

$$u^{(n+1)} = u^{(n)} + \alpha^{(n)} p^{(n)}$$

satisfies the following condition:

$$\|f - Ku^{(n+1)}\|_{\ell_2} = \min_{v \in u^{(n)} + \operatorname{span}(p^{(n)})} \|f - Kv\|_{\ell_2}.$$

Hint: The function $q(\alpha) = \|f - K\left[u^{(n)} + \alpha\, p^{(n)}\right]\|_{\ell_2}^2$ is a simple (quadratic) function in α.

27. Let K be a symmetric and positive definite matrix, let $r^{(n)}$ be the residual of the approximation $u^{(n)}$ of u, the exact solution of the linear system

$$Ku = f,$$

and let $p^{(n-1)}$ be a given search direction (of the previous step). Determine the new search direction $p^{(n)}$ of the form

$$p^{(n)} = r^{(n)} + \beta^{(n-1)} p^{(n-1)},$$

such that

$$(Kp^{(n)}, Kp^{(n-1)})_{\ell_2} = 0.$$

by appropriately choosing the parameter $\beta^{(n-1)}$.

28. Let K be a symmetric and positive definite matrix. The so-called CR method (conjugate residual method) is given by

initialization: $r^{(0)} = b - Ku^{(0)}$. For $n = 0, 1, 2, \ldots$:

$$\begin{aligned} p^{(n)} &= \begin{cases} r^{(0)} & \text{for} \quad n = 0 \\ r^{(n)} + \beta^{(n-1)} p^{(n-1)} & \text{for} \quad n \geq 1 \end{cases} \\ u^{(n+1)} &= u^{(n)} + \alpha^{(n)} p^{(n)} \\ r^{(n+1)} &= r^{(n)} - \alpha^{(n)} K p^{(n)}, \end{aligned}$$

where the parameters $\beta^{(n-1)}$ and $\alpha^{(n)}$ are chosen such that

$$(Kp^{(n)}, Kp^{(n-1)})_{\ell_2} = 0$$

and

$$\|f - Ku^{(n+1)}\|_{\ell_2} = \min_{v \in u^{(n)} + \mathrm{span}(p^{(n)})} \|f - Kv\|_{\ell_2}.$$

Show:

$$(Kr^{(n+1)}, p^{(n)})_{\ell_2} = 0$$

and

$$(Kr^{(n+1)}, r^{(n)})_{\ell_2} = 0.$$

(This last property motivates the name of the method.)

29. Let K be a symmetric and positive definite matrix. Formulate the CG method for solving the linear system

$$Ku = f,$$

but use the scalar product $(w, v)_K \equiv (Kw, v)_{\ell_2}$ instead of the originally used scalar product $(w, v)_{\ell_2}$. Show that this method coincides with the CR method.

30. Let K be a symmetric and positive definite matrix and let

$$J_K(v) = \frac{1}{2}(Kv, v)_K - (f, v)_K$$

Express $J_K(v)$ in terms of $\|f - Kv\|_{\ell_2}^2$ and $\|f\|_{\ell_2}^2$.

31. Let K be a symmetric and positive definite matrix. Show that the CR method is a Krylov subspace method with

$$\|f - Kv^{(n)}\|_{\ell_2} = \min_{v \in u^{(0)} + \mathcal{K}_n(K, r^{(0)})} \|f - Kv\|_{\ell_2}.$$

Hint: The CR method can be interpreted as a CG method with a different scalar product.