ÜBUNGEN ZU

NUMERIK PARTIELLER DIFFERENTIALGLEICHUNGEN

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7. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\Gamma = \Gamma_D \cup \Gamma_N$. Find the variational formulation ($V = ?$, $V_0 = ?$, $V_g = ?$, $a = ?$, $F = ?$) for the following boundary value problem:

$$\begin{aligned} -\Delta u(x) &= f(x) \quad x \in \Omega, \\ u(x) &= g_D(x) \quad x \in \Gamma_D, \\ \frac{\partial u}{\partial n}(x) &= g_N(x) \quad x \in \Gamma_N. \end{aligned}$$

8. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ. The scalar product in the Hilbert space $H = L^2(\Omega)$ is given by

$$(w, v)_0 = \int_\Omega w(x)v(x)\, dx.$$

The scalar product in the Hilbert space $V = H^1(\Omega)$ is given by

$$(w, v)_1 = (w, v)_0 + (\operatorname{grad} w, \operatorname{grad} v)_0 = \int_\Omega w(x)v(x)\, dx + \int_\Omega \operatorname{grad} w(x) \cdot \operatorname{grad} v(x)\, dx.$$

Let $f \in L^2(\Omega)$. Consider the variational problem:

Find $u \in V$ such that

$$(u, v)_1 = (f, v)_0 \quad \text{for all } v \in V.$$

What is the corresponding boundary value problem in classical formulation?

9. Let $\Omega = (0, 1) \times (0, 1)$ and $\Gamma_D = \{0\} \times [0, 1]$. Show the Friedrichs inequality: There is a constant $C_P > 0$ with

$$\|v\|_0 \le C_P\, |v|_1 \quad \text{for all } v \in V_0 = \{w \in H^1(0, 1) : w(x) = 0 \text{ on } \Gamma_D\}.$$

10. To each real n-by-n matrix A a bilinear form a, given by

$$a(w, v) = (Aw, v)_{\ell_2} \quad \text{for all } w, v \in \mathbb{R}^n,$$

can be uniquely associated and vice versa. Here, $(.,.)_{\ell_2}$ denotes the Euclidean scalar product with corresponding Euclidean norm $\|.\|_{\ell_2}$. The smallest possible bound μ_2, which satisfies the condition

$$|a(w, v)| \le \mu_2\, \|w\|_{\ell_2} \|v\|_{\ell_2} \quad \text{for all } w, v \in \mathbb{R}^n$$

is obviously given by the quantity

$$\sup_{0\neq w\in\mathbb{R}^n}\sup_{0\neq v\in\mathbb{R}^n}\frac{a(w,v)}{\|w\|_{\ell_2}\|v\|_{\ell_2}}.$$

Show that

$$\sup_{0\neq w\in\mathbb{R}^n}\sup_{0\neq v\in\mathbb{R}^n}\frac{a(w,v)}{\|w\|_{\ell_2}\|v\|_{\ell_2}}=\sigma_{\max}(A),$$

where $\sigma_{\max}(A)$ denotes the largest singular value of A.

Hint: Without proof you can use that each real matrix A can be written in the following form: $A = U\Sigma V^T$ with orthogonal matrices U and V and a diagonal matrix Σ, whose diagonal elements are the (non-negative) singular values. Replace w by $V\widetilde{w}$ and v by $U\widetilde{v}$ and observe that $\|w\|_{\ell_2} = \|\widetilde{w}\|_{\ell_2}$ and $\|v\|_{\ell_2} = \|\widetilde{v}\|_{\ell_2}$.

11. Assume the notations in the previous example. The largest possible bound μ_1, which satisfies the condition

$$a(v,v) \geq \mu_1\,\|v\|_{\ell_2}^2 \quad \text{for all } v\in\mathbb{R}^n$$

is obviously given by the quantity

$$\inf_{0\neq v\in\mathbb{R}^n}\frac{a(v,v)}{\|v\|_{\ell_2}^2}.$$

Show that

$$\inf_{0\neq v\in\mathbb{R}^n}\frac{a(v,v)}{\|v\|_{\ell_2}^2}=\lambda_{\min}(A_s),$$

where $\lambda_{\min}(A_s)$ denotes the smallest eigenvalue of the (symmetric) matrix $A_s = (A + A^T)/2$.

Hint: Show that $(Av, v)_{\ell_2} = (A_s v, v)_{\ell_2}$ and use the representation $A_s = UDU^T$ of the symmetric matrix A_s with an orthogonal matrix U and the diagonal matrix D, whose diagonal elements are the eigenvalues of A_s.

12. Assume the notations in the two previous examples. Let A be a (not necessarily symmetric) real n-by-n matrix with $(Av, v)_{\ell_2} \geq 0$ for all $v \in \mathbb{R}^n$. Derive a linear system of equations which characterizes a solution $u \in \mathbb{R}^n$ of the minimization problem

$$J(u) = \min_{v\in\mathbb{R}^n} J(v)$$

with

$$J(v) = \frac{1}{2}(Av, v)_{\ell_2} - (f, v)_{\ell_2}$$

for a given $f \in \mathbb{R}^n$.